Math 127 **Final Review** **Name**______________________________

No calculator, unless marked with an asterisk (*)

Final Exam Formulas			
$\vec{v} = \lVert\vec{v}\rVert cos\theta \cdot \boldsymbol{i} + \lVert\vec{v}\rVert sin\theta \cdot \boldsymbol{j}$	$\cos\theta = \frac{v \cdot w}{\lVert v\rVert \lVert w\rVert}$	$Proj_w v = \frac{\vec{v} \cdot \vec{w}}{\lVert\vec{w}\rVert^2}(\vec{\boldsymbol{w}})$	$\boldsymbol{v}_2 = \boldsymbol{v} - \boldsymbol{v}_1$

1) Write the equation of an ellipse in standard form that meets the requirements below:
foci: $(0, -2), (0, 2)$; y-intercepts: -5 and 5.

Note: graphs of conic sections for problems in this packet were made with the Algebra (Main) App, available at: www.mathguy.us/PCApps.php.

This ellipse has foci $(0, \pm 2)$, which are on the y-axis.
Therefore, the ellipse has a vertical major axis.

The standard form for an ellipse with a vertical major axis is:

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

Remember that $a > b$ for an ellipse. That's why a^2 is in the denominator of the y-term.

The values of a and b can be determined from the foci and the y-intercepts.

The center of the ellipse is halfway between the foci, i.e., at $(0, 0)$, so $(h, k) = (0, 0)$.

The foci are located $c = 2$ units from the center. So, we have determined that:

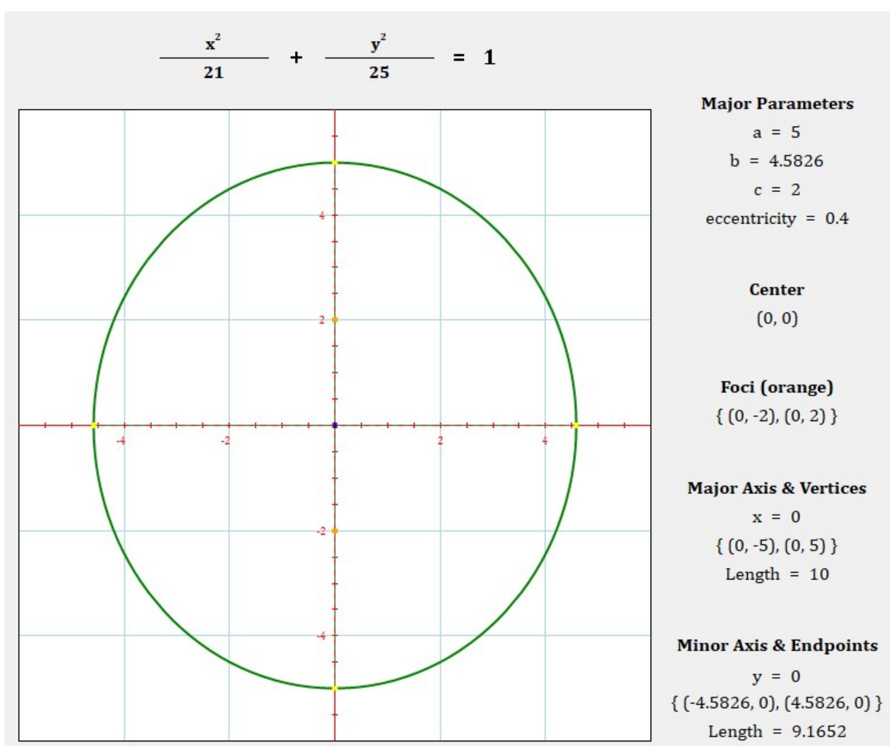
- $h = 0,\ k = 0,\ c = 2$

The y-intercepts, then, are major axis vertices, which are located $a = 5$ units from the center. So, we have determined that:

- $a = 5 \quad \Rightarrow \quad a^2 = 25$
- $c^2 = a^2 - b^2$ for an ellipse, so: $2^2 = 5^2 - b^2$, giving $b^2 = 21$

Then, substituting values into the standard form equation gives:

$$\boldsymbol{\frac{x^2}{21} + \frac{y^2}{25} = 1}$$



2) Write the equation of a hyperbola in standard form that meets the requirements below:
foci: $(0,-4),(0,4)$; vertices: $(0,-3),(0,3)$.

This hyperbola has foci $(0,\pm 4)$, and therefore has a vertical transverse axis.

The standard form for a hyperbola with a vertical transverse axis is:

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

Remember that a^2 is in the denominator of the lead term for a hyperbola.

The values of a and b can be determined from the foci and the vertices.

The center of the hyperbola is halfway between the foci, i.e., at $(0,0)$, so $(h,k)=(0,0)$.

The foci are located $c=4$ units from the center. So, we have determined that:

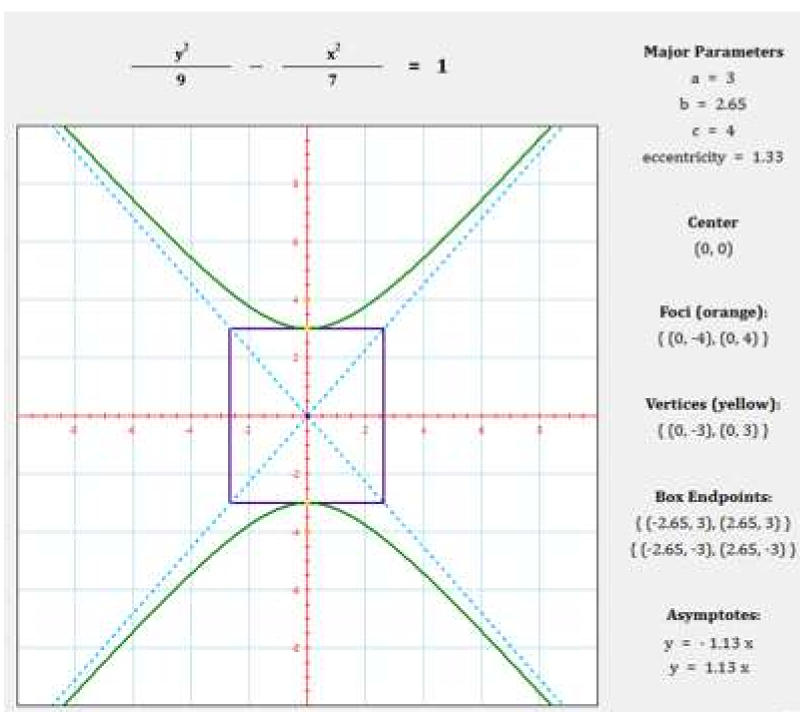
- $h=0,\ k=0,\ c=4$

The vertices are located $a=3$ units from the center. So, we have determined that:

- $a=3 \quad \Rightarrow \quad a^2=9$
- $c^2=a^2+b^2$ for a hyperbola, so $4^2=3^2+b^2$, giving $b^2=7$

Then, substituting values into the standard form equation gives:

$$\frac{y^2}{9}-\frac{x^2}{7}=1$$



For #3 – 5, graph the conic and find the requested information. If needed, round to 3 decimal places.

*3) $4x^2 + 8x + 9y^2 = 32$ (Convert to standard form by completing the square)

$4x^2 + 8x + 9y^2 = 32$

$(4x^2 + 8x + \quad) + 9y^2 = 32$

$4(x^2 + 2x + \quad) + 9y^2 = 32$

$4(x^2 + 2x + 1) + 9y^2 = 32 + 4(1)$

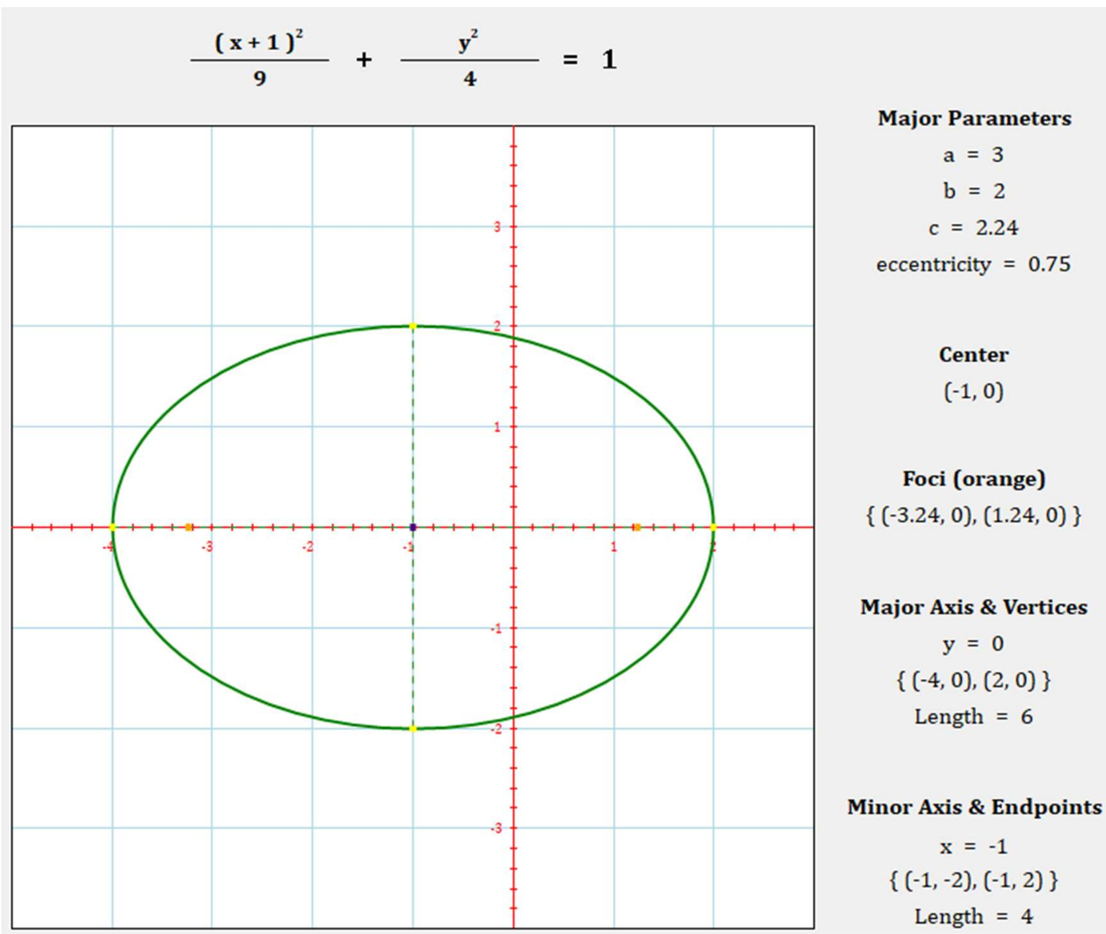
$4(x+1)^2 + 9y^2 = 36$

$\frac{(x+1)^2}{9} + \frac{y^2}{4} = 1$

Center: $(-1, 0)$

The foci are $\sqrt{9-4} = \sqrt{5} \sim 2.236$ away from the center along the major axis, i.e., in both x-directions (left and right).

So, the **foci** are: $(-1 - 2.236, 0) = (-3.236, 0)$ and $(-1 + 2.236, 0) = (1.236, 0)$.



4) $x^2 - 2x + 7y - 34 = 0$ (Convert to standard form by completing the square)

The key to graphing a parabola is to identify its vertex and orientation (which way it opens). Consider the form of the above equation:

$$(x-h)^2 = 4p(y-k)$$

> The term "latus rectum" is Latin, and means "straight side". Although we only study the latus rectum of the parabola in high school, the ellipse and hyperbola each have two latus recta.

$x^2 - 2x + 7y - 34 = 0$

$(x^2 - 2x + \_\_\_\_) = -7y + 34$

$(x^2 - 2x + 1) = -7y + 34 + 1$

$\mathbf{(x-1)^2 = -7(y-5)}$

From this equation, we can determine the following:

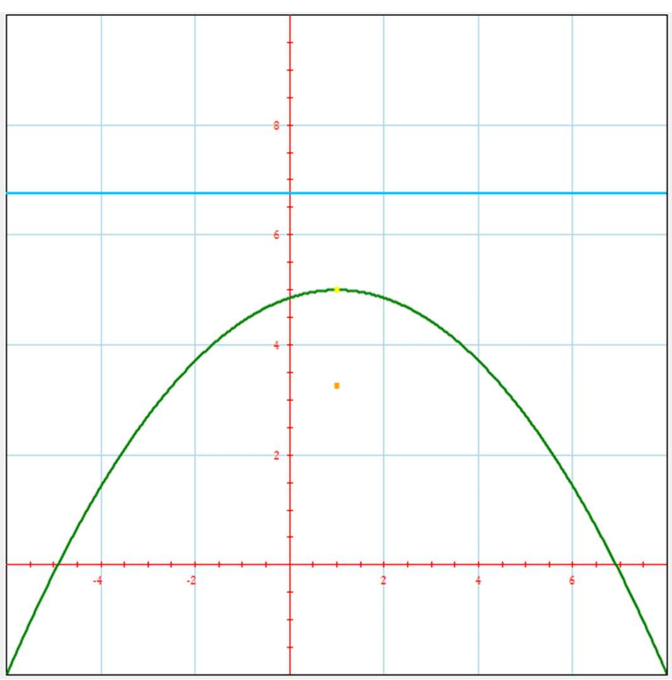
- The **vertex** of the parabola is $(h, k) = (\mathbf{1}, \mathbf{5})$.
- Since the x-term is squared, the parabola has a horizontal Directrix (i.e., it opens up or down).
- The **length of the latus rectum** is $|4p| = \mathbf{7}$. $p = -7/4$ is negative, so the parabola opens down.
- Since the x-term is squared, the **focus** is located $7/4$ units below the vertex, $(1, 5)$: $(\mathbf{1}, \mathbf{3.25})$.
- The **equation of the directrix** is: $y = k - p \quad \Rightarrow \quad y = 5 - \left(-\frac{7}{4}\right) \quad \Rightarrow \quad \mathbf{y = 6.75}$

Let's find a couple of points to help us draw our graph of the parabola. Rewrite the equation in a simpler form to find y, given x.

$$y = -\frac{1}{7}(x-1)^2 + 5$$

We already have a point – the vertex, at $(1, 5)$. Let's find a couple more:

- Let $x = -2$. Then $-\frac{1}{7}(-2-1)^2 + 5 = 3\frac{5}{7}$. This gives us the point $\left(-2, 3\frac{5}{7}\right)$.
- Let $x = 4$. Then $-\frac{1}{7}(4-1)^2 + 5 = 3\frac{5}{7}$. This gives us the point $\left(4, 3\frac{5}{7}\right)$.



Major Parameters
a = -0.143
p = -1.75
eccentricity = 1

Vertex (yellow point)
(1, 5)

Focus (orange point)
(1, 3.25)

Directrix (blue line)
y = 6.75

Axis of Symmetry
x = 1

5) $\frac{(x-2)^2}{16}-\frac{(y+2)^2}{4}=1$

This equation is already in standard form.

$\frac{(x-2)^2}{16}-\frac{(y+2)^2}{4}=1$ Standard form is: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ or $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$

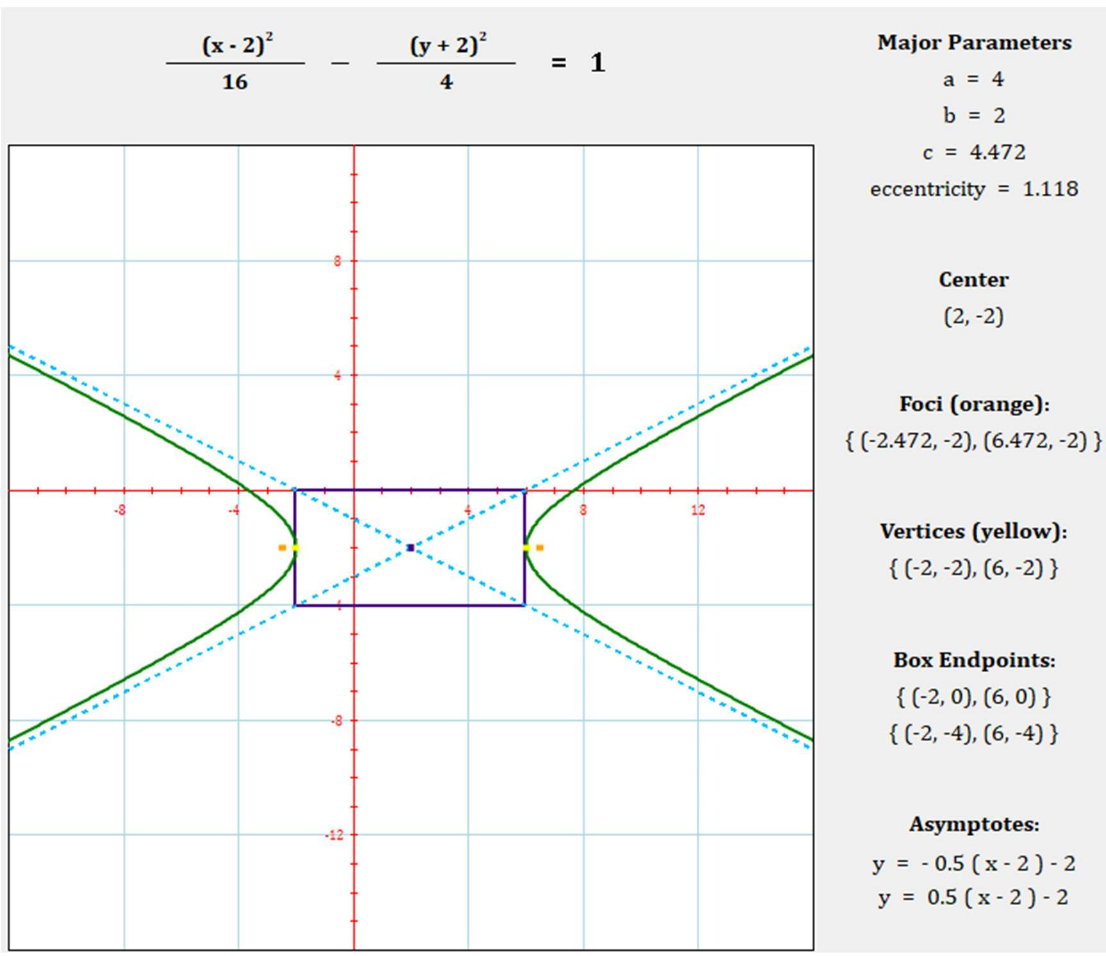
Since the x-term is positive, we know that the hyperbola has a horizontal transverse axis.

- $h=2,\ k=-2$
- $a^2=16$, so $a=4$, $b^2=4$, so $b=2$, $c^2=a^2+b^2=16+4=20$, so $c=\sqrt{20}\sim 4.472$
- **The center is** (h,k): $(\mathbf{2},-\mathbf{2})$
- The center, vertices, and foci lie on the horizontal transverse axis (HTA). On an HTA:
  The vertices are a units left and right from the center: $(2\pm 4,-2)\ \Rightarrow\ \{(-2,-2),(6,-2)\}$
  The foci are c units left and right from the center: $(2\pm\sqrt{20},-2)\ \Rightarrow\ \{(-\mathbf{2.472},-\mathbf{2}),(\mathbf{6.472},-\mathbf{2})\}$

The asymptotes are not required, but for a hyperbola with a horizontal transverse axis are: $y=\pm\frac{b}{a}(x-h)+k$.

The asymptotes are: $y=\pm\frac{b}{a}(x-h)+k\ \Rightarrow\ y=\pm\frac{1}{2}(x-2)-2$

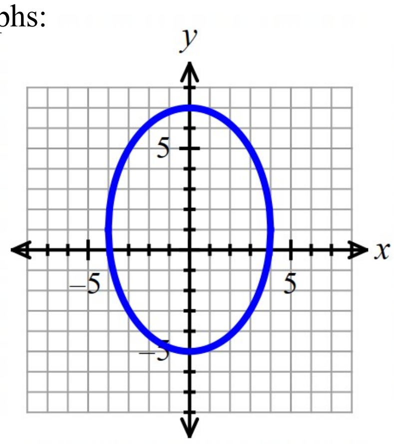
To graph the hyperbola, first graph the asymptotes and the vertices, and then sketch in the rest.



For #6 – 11, match each graph to its equation. No item will be used more than once. Not all equations will be used.

Graphs:

6)



An **ellipse** has a $+$ sign between terms and different denominators in the two terms.

The center is $(0, 1)$.

Each denominator is the squares of half of the length of its axis.

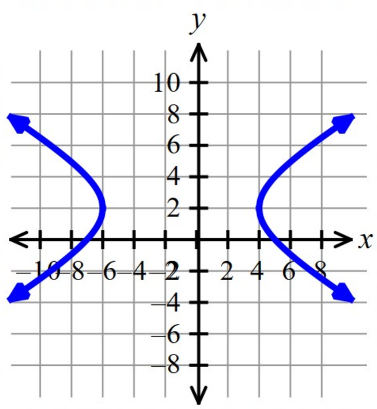
x-axis length is 8, so the x-denominator is $\left(\frac{8}{2}\right)^2 = 16$.

y-axis length is 12, so the y-denominator is $\left(\frac{12}{2}\right)^2 = 36$.

The equation, then, is:

$$\frac{x^2}{16} + \frac{(y-1)^2}{36} = 1$$ **Answer D**

7)



A **hyperbola** has a $-$ sign between terms.

It opens left and right if the x-term is positive.

Its vertices are $(-6, 2), (4, 2)$.

Its center is halfway between the vertices: $(-1, 2)$.

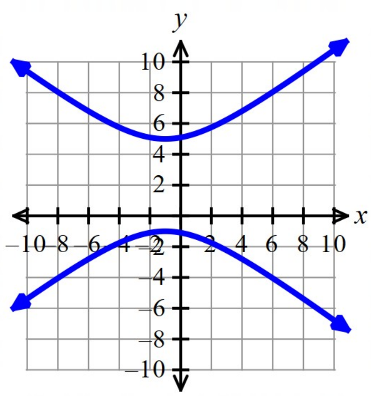
Its horizontal transverse axis has length: $4 - (-6) = 10$. The value of a is half this length, $a = 5$, and so $a^2 = 25$.

This hyperbola, then, has an equation of the form:

$$\frac{(x+1)^2}{25} - \frac{(y-2)^2}{b^2} = 1$$

Answers A and B are hyperbolas. **Answer B** has the proper form.

8)



A **hyperbola** has a $-$ sign between terms.

It opens up and down if the y-term is positive.

Its vertices are $(-1, -5), (-1, 1)$.

Its center is halfway between the vertices: $(-1, 2)$.

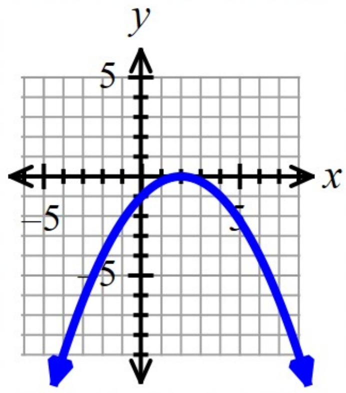
Its vertical transverse axis has length: $1 - (-5) = 6$. The value of a is half this length, $a = 3$, and so $a^2 = 9$.

This hyperbola, then, has an equation of the form:

$$\frac{(y-2)^2}{9} - \frac{(x+1)^2}{b^2} = 1$$

Answers A and B are hyperbolas. **Answer A** has the proper form.

9)



A **parabola** has only one of the terms squared.

This parabola opens down (in the y-direction), so p is negative and the squared term is the x-term.

The vertex is $(2, 0)$.

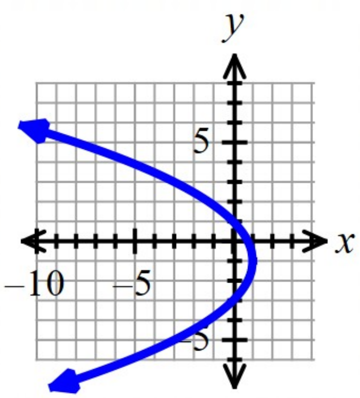
The equation must have the form:

$(\boldsymbol{x-2})^2 = -\boldsymbol{cy}$, with c a constant.

Answers C and F are parabolas.

Answer F has the proper form.

10)



A **parabola** has only one of the terms squared.

This parabola opens to the left (in the x-direction), so p is negative and the squared term is the y-term.

The vertex is $(1, -1)$.

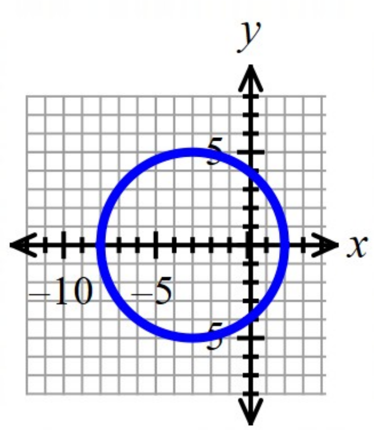
The equation must have the form:

$(\boldsymbol{y+1})^2 = -\boldsymbol{c}(\boldsymbol{x-1})$, with c a constant.

Answers C and F are parabolas.

Answer C has the proper form.

11)



A **circle** has a $+$ sign between terms, and no (or the same) denominators.

The center of this circle is $(-3, 0)$.

The radius of the circle is $r = 5$, so $r^2 = 25$.

The equation must have the form:

$(\boldsymbol{x+3})^2 + \boldsymbol{y}^2 = \boldsymbol{25}$.

Answers F, G, and H are circles.

Answer H is correct.

12) Write the equation of a parabola in standard form that meets the requirements below:
Focus at $(3, -2)$ and directrix at $y = 0$.

The parabola described above has a horizontal Directrix, so it opens up or down.

The focus is below the Directrix, so the parabola opens down. *Note: parabolas always open away from the Directrix, toward the focus*.

For a parabola with a horizontal Directrix:

- The vertex is halfway between the focus and the Directrix, so the vertex is:

$$(h, k) = \left(3, \frac{-2 + (0)}{2}\right) = (3, -1) \quad \Rightarrow \quad h = 3; \; k = -1$$

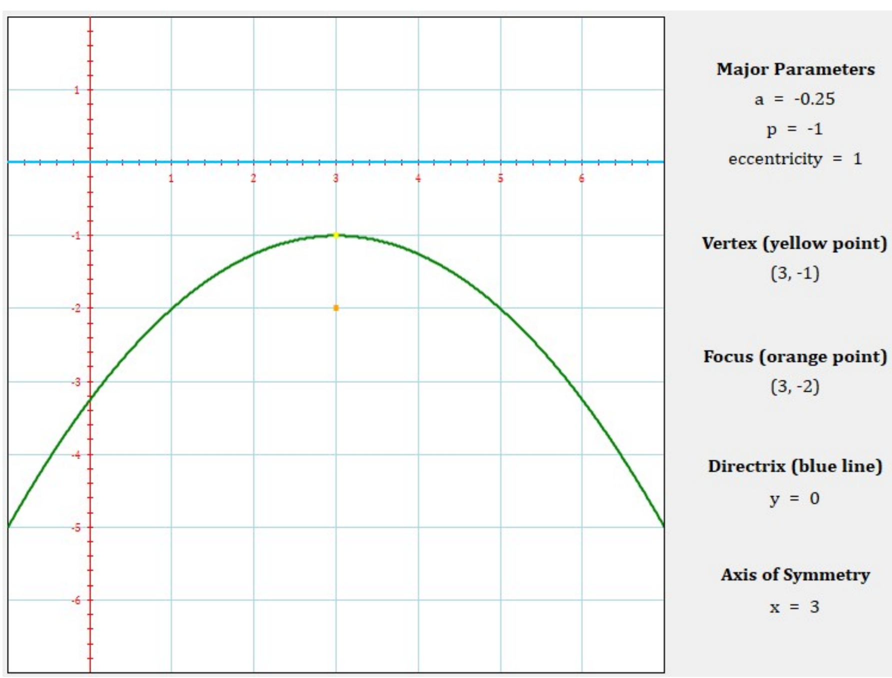
The focus is p units away from the vertex: $p = (focus\ y\ value) - (vertex\ y\ value)$

- $p = -2 - (-1) = -1, \; 4p = -4$

We can now write the equation in standard form: $(x - h)^2 = 4p(y - k)$

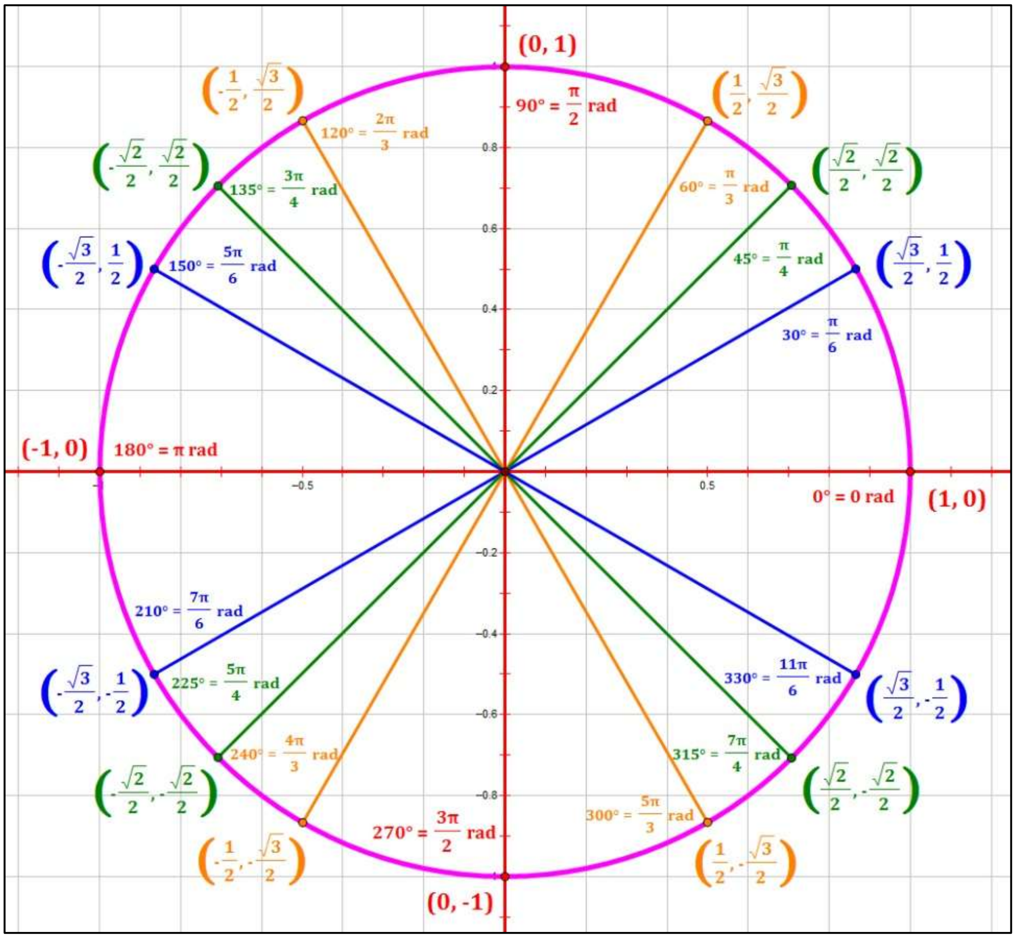
$$\mathbf{(x - 3)^2 = -4(y + 1)}$$

Although a graph is not required, here's what it would look like:

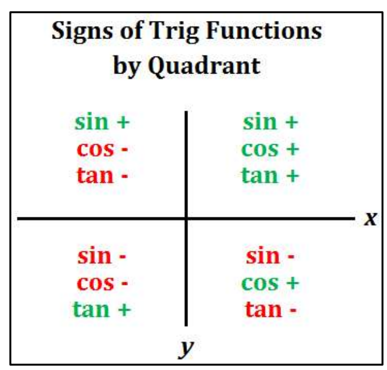


Math 127 **Final Review** **Name______________________________**

For Trigonometry questions, a couple of things to help out:



Trig Functions of Special Angles (θ)				
Radians	**Degrees**	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	**0°**	$\frac{\sqrt{0}}{2}=0$	$\frac{\sqrt{4}}{2}=1$	$\frac{\sqrt{0}}{\sqrt{4}}=0$
$\pi/6$	**30°**	$\frac{\sqrt{1}}{2}=\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{\sqrt{3}}=\frac{\sqrt{3}}{3}$
$\pi/4$	**45°**	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{\sqrt{2}}=1$
$\pi/3$	**60°**	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2}=\frac{1}{2}$	$\frac{\sqrt{3}}{\sqrt{1}}=\sqrt{3}$
$\pi/2$	**90°**	$\frac{\sqrt{4}}{2}=1$	$\frac{\sqrt{0}}{2}=0$	*undefined*



For 13 – 16, verify the trig identity:

13) $\tan x(\cot x - \cos x) = 1 - \sin x$

$$\tan x \cdot (\cot x - \cos x) \qquad = 1 - \sin x$$

$$\frac{\sin x}{\cos x} \cdot \left(\frac{\cos x}{\sin x} - \frac{\cos x}{1}\right)$$

$$\left(\frac{\sin x}{\cos x} \cdot \frac{\cos x}{\sin x}\right) - \left(\frac{\sin x}{\cos x} \cdot \frac{\cos x}{1}\right)$$

$$\mathbf{1 \quad - \quad \sin x \qquad = 1 - \sin x} \qquad \checkmark$$

14) $\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$

$$\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} \qquad = \tan\alpha + \tan B$$

$$\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}$$

$$\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}$$

$$\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}$$

$$\mathbf{\tan\alpha + \tan B \qquad = \tan\alpha + \tan B} \qquad \checkmark$$

15) $\frac{1+\cos}{\sin 2x} = \cot x$

$$\frac{1 + \cos 2x}{\sin 2x} \qquad = \cot x$$

$$\frac{1 + (2\cos^2 x - 1)}{2\sin x\cos x}$$

$$\frac{2\cos^2 x}{2\sin x\cos x}$$

$$\frac{\cos x}{\sin x}$$

$$\mathbf{\cot x \qquad = \cot x} \qquad \checkmark$$

16) $\sin\left(x+\frac{\pi}{2}\right)=\cos x$

$$\sin\left(x+\frac{\pi}{2}\right) \qquad = \cos x$$

$$\sin x\cos\frac{\pi}{2}+\cos x\sin\frac{\pi}{2}$$

$$(\sin x)\cdot 0+(\cos x)\cdot 1$$

$$\mathbf{\cos x} \qquad \mathbf{=\cos x} \qquad \checkmark$$

17) Verify the identity: $\tan^2\theta+4=\sec^2\theta+3$

$$\tan^2\theta+4 \qquad =\sec^2\theta+3$$

$$(\tan^2\theta+1)+3$$

$$\mathbf{\sec^2\theta+3} \qquad \mathbf{=\sec^2\theta+3} \qquad \checkmark$$

For 18 – 19: Find the exact value by using a sum or difference identity (Hint: consider using a sum or difference identity).

18) $\tan\frac{7\pi}{12}$

$$\tan\frac{7\pi}{12}=\tan\left(\frac{3\pi}{12}+\frac{4\pi}{12}\right)=\tan\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$$

$$=\frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{3}}{1-\left(\tan\frac{\pi}{4}\right)\left(\tan\frac{\pi}{3}\right)}$$

$$=\frac{1+\sqrt{3}}{1-1\cdot\sqrt{3}}=\frac{1+\sqrt{3}}{1-\sqrt{3}}\cdot\frac{1+\sqrt{3}}{1+\sqrt{3}}=\frac{1+2\sqrt{3}+3}{1-3}=\frac{4+2\sqrt{3}}{-2}=\mathbf{-2-\sqrt{3}}$$

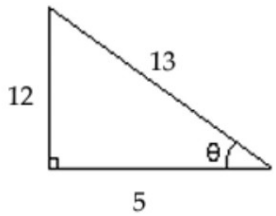
19) $\cos\frac{11\pi}{12}$

$$\cos\frac{11\pi}{12}=\cos\left(\frac{3\pi}{12}+\frac{8\pi}{12}\right)=\cos\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)$$

$$=\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{2\pi}{3}\right)-\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{2\pi}{3}\right)$$

$$=\frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)=\mathbf{\frac{-\sqrt{2}-\sqrt{6}}{4}}=\mathbf{-\frac{\sqrt{2}+\sqrt{6}}{4}}$$

20) Use the figure to find the exact value of $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$.



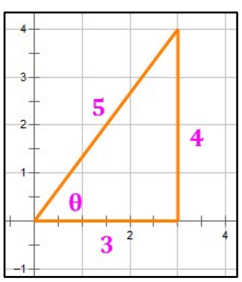
$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{12}{13}\cdot\frac{5}{13} = \mathbf{\frac{120}{169}}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2 = \frac{25-144}{169} = \mathbf{\frac{-119}{169}}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{120}{169}}{\frac{-119}{169}} = -\mathbf{\frac{120}{119}}$$

21) Use the given information to find the exact value of $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$.

$\sin\theta = \frac{4}{5}$, θ lies in quadrant I



$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{4}{5}\cdot\frac{3}{5} = \mathbf{\frac{24}{25}}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = -\mathbf{\frac{7}{25}}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = -\mathbf{\frac{24}{7}}$$

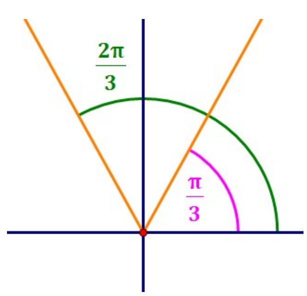
For 22 – 23, find all solutions of the following equations.

22) $2\sin x - \sqrt{3} = 0$

$$2\sin x - \sqrt{3} = 0$$

$$2\sin x = \sqrt{3}$$

$$\sin x = \frac{\sqrt{3}}{2}$$



The drawing at left illustrates the two angles in $[0, 2\pi)$ for which $\sin x = \frac{\sqrt{3}}{2}$. To get all solutions, we need to add all integer multiples of 2π to these solutions. So,

$$\boldsymbol{x \in \left\{\frac{\pi}{3} + 2n\pi\right\} \cup \left\{\frac{2\pi}{3} + 2n\pi\right\}}$$

23) $\tan x \sec x = -2 \tan x$

$\tan x \sec x = -2 \tan x$

$\tan x \sec x + 2 \tan x = 0$

$\tan x\,(\sec x + 2) = 0$

$\tan x = 0$ or $(\sec x + 2) = 0$

$x = 0 + n\pi = n\pi$

$(\sec x + 2) = 0$

$\sec x = -2$

$\cos x = -\frac{1}{2}$

$x = \frac{2\pi}{3} + 2n\pi$ or $x = \frac{4\pi}{3} + 2n\pi$

Collecting the various solutions, $\boldsymbol{x \in \{n\pi\} \cup \left\{\frac{2\pi}{3} + 2n\pi\right\} \cup \left\{\frac{4\pi}{3} + 2n\pi\right\}}$

Note: the solution involving the tangent function has two solutions in the interval $[0, 2\pi)$. These solutions are π radians apart, as are most solutions involving the tangent function. Therefore, we can simplify the answers by showing only one base solution and adding $n\pi$, instead of showing two base solutions that are π apart, and adding $2n\pi$ to each.

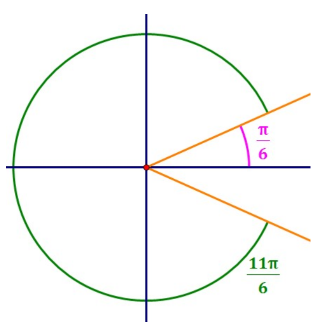
For example, the following two solutions for $\tan x = 0$ are combined into the single solution given above:

$$\left.\begin{array}{l} x = 0 + 2n\pi = \{\ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots\} \\ x = \pi + 2n\pi = \{\ldots, -3\pi, -\pi, \pi, 3\pi, 5\pi \ldots\} \end{array}\right\} \quad x = 0 + n\pi = \{\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\}$$

For 24 – 27, solve the equation on the interval $[0, 2\pi)$.

24) $\cos 2x = \frac{\sqrt{3}}{2}$

We want all solutions to $\cos u = \frac{\sqrt{3}}{2}$ where $u = 2x$ is an angle in the interval $[0, 4\pi)$. Note that, beyond the two solutions suggested by the diagram, additional solutions are obtained by adding 2π to those two solutions.



Note that there are 4 solutions because the usual number of solutions (i.e., 2) is increased by a factor equal to the coefficient of x, i.e., 2.

Using the diagram at left, we get the following solutions:

$$u = 2x = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$$

Then, dividing by 2, we get:

$$\boldsymbol{x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}}$$

We cannot simplify these solutions any further.

25) $\cos^2 x + 2\cos x + 1 = 0$

The trick to this problem is to replace the trigonometric function, in this case, $\cos x$, with a variable, like u, that will make it easier to see how to factor the expression. If you can see how to factor the expression without the trick, by all means proceed without it.

Let $u = \cos x$, and our equation becomes: $u^2 + 2u + 1 = 0$.

This equation factors to get: $(u+1)^2 = 0$

Substituting $\cos x$ back in for u gives: $(\cos x + 1)^2 = 0$

And finally: $\cos x + 1 = 0 \quad \Rightarrow \quad \cos x = -1$

The only solution for this on the interval $[0, 2\pi)$ is: $\boldsymbol{x = \pi}$

26) $\cos x + 2\cos x \sin x = 0$

$\cos x + 2\cos x \sin x = 0$

$\cos x\,(1 + 2\sin x) = 0$ → $(1 + 2\sin x) = 0$

$\cos x = 0$ or $(1 + 2\sin x) = 0$ $\sin x = -\frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

Collecting the various solutions, $\boldsymbol{x = \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}}$

27) $\cos\left(x + \frac{\pi}{3}\right) + \cos\left(x - \frac{\pi}{3}\right) = 1$

The following formulas will help us solve this problem.

$\cos\,(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ $\cos\,(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos\left(x + \frac{\pi}{3}\right) + \cos\left(x - \frac{\pi}{3}\right) = 1$

$\cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3} + \cos x\cos\frac{\pi}{3} + \sin x\sin\frac{\pi}{3} = 1$

$2\cos x\cos\frac{\pi}{3} = 1$

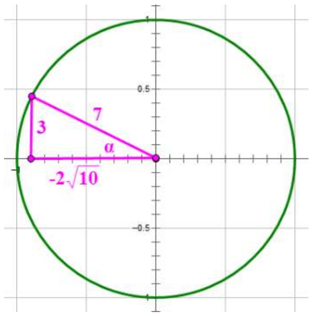
$2\cos x \cdot \frac{1}{2} = 1$

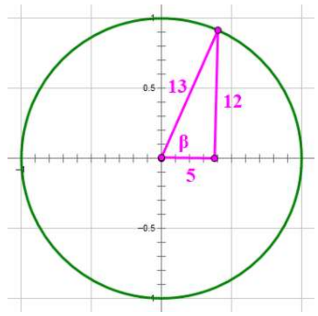
$\cos x = 1$

$\boldsymbol{x = 0}$

28) Given $\sin\alpha = \frac{3}{7}$, α lies in quadrant II, $\cos\beta = \frac{5}{13}$, and β lies in quadrant I. Find $\sin(\alpha - \beta)$.

Construct triangles for the two angles, being careful to consider the signs of the values in each quadrant:





Then,

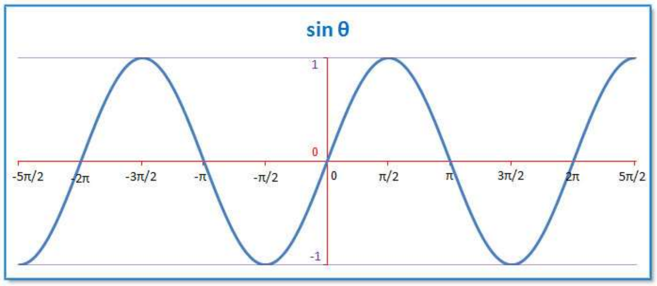
$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha)$$

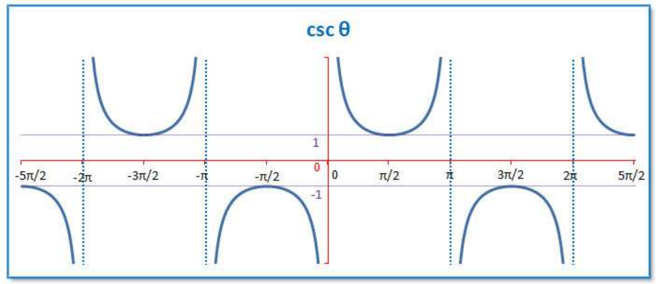
$$= \left(\frac{3}{7} \cdot \frac{5}{13}\right) - \left(\frac{12}{13} \cdot \frac{-2\sqrt{10}}{7}\right)$$

$$= \frac{\mathbf{15+24\sqrt{10}}}{\mathbf{91}}$$

Trigonometry Graphing Questions are next.

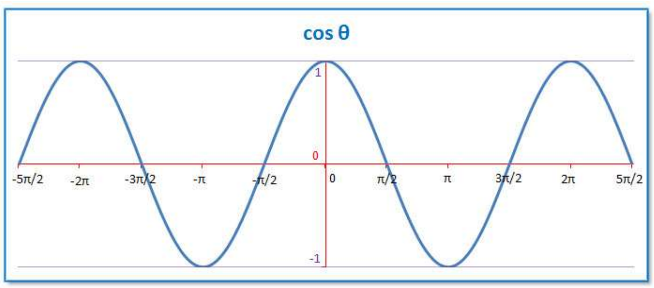
Graphs of Basic (Parent) Trigonometric Functions

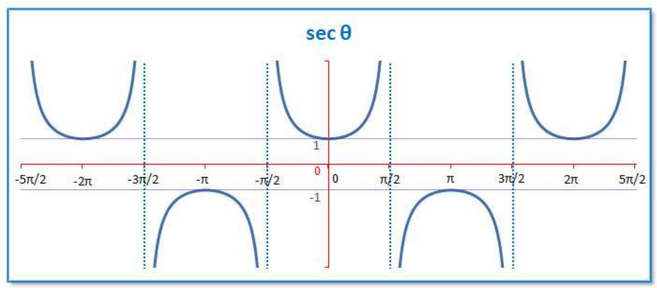




The sine and cosecant functions are reciprocals. So:

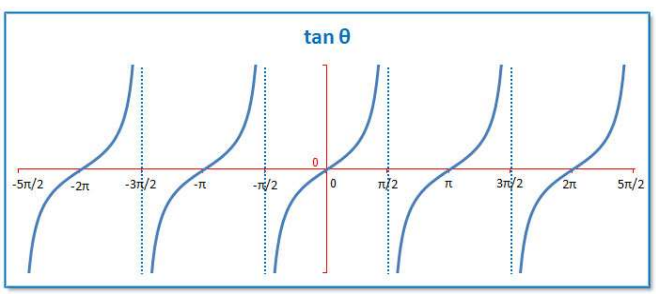
$$\sin\theta = \frac{1}{\csc\theta} \quad \text{and} \quad \csc\theta = \frac{1}{\sin\theta}$$

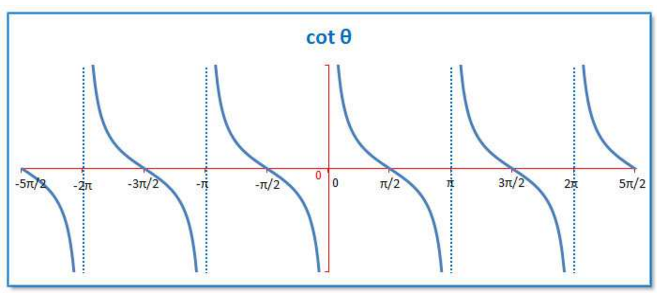




The cosine and secant functions are reciprocals. So:

$$\cos\theta = \frac{1}{\sec\theta} \quad \text{and} \quad \sec\theta = \frac{1}{\cos\theta}$$





The tangent and cotangent functions are reciprocals. So:

$$\tan\theta = \frac{1}{\cot\theta} \quad \text{and} \quad \cot\theta = \frac{1}{\tan\theta}$$

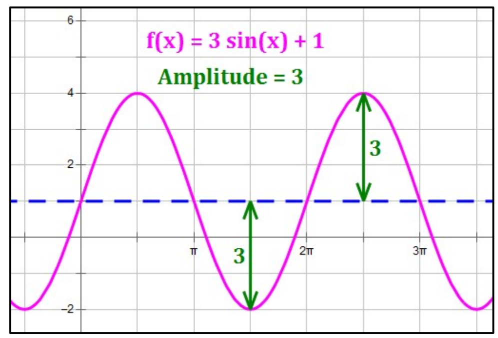
Characteristics of Trigonometric Function Graphs

All trigonometric functions are periodic, meaning that they repeat the pattern of the curve (called a cycle) on a regular basis. The key characteristics of each curve, along with knowledge of the parent curves are sufficient to graph many trigonometric functions. Let's consider the general function:

$$f(x) = \mathrm{A} \cdot trig(\mathrm{B}x - \mathrm{C}) + \mathrm{D}$$

where A, B, C and D are constants and "$trig$" is any of the six trigonometric functions (sine, cosine, tangent, cotangent, secant, cosecant).

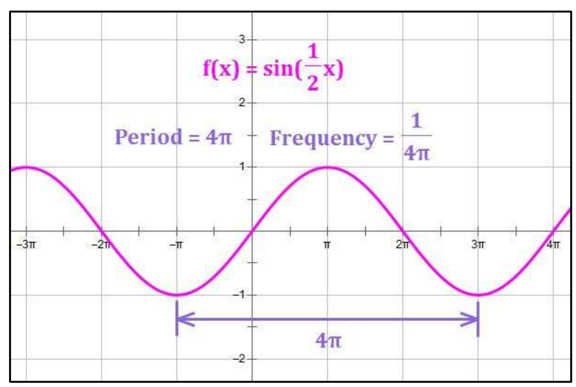
Amplitude



Amplitude is the measure of the distance of peaks and troughs from the midline (i.e., center) of a *sine or cosine function*; amplitude is always positive. The other four functions do not have peaks and troughs, so they do not have amplitudes. For the general function, $f(x)$, defined above, $\text{amplitude} = |\mathrm{A}|$.

Period

Period is the horizontal width of a single cycle or wave, i.e., the distance it travels before it repeats. Every trigonometric function has a period. The periods of the *parent functions* are as follows: for sine, cosine, secant and cosecant, $\text{period} = 2\pi$; for tangent and cotangent, $\text{period} = \pi$.



For the general function, $f(x)$, defined above,

$$\text{period} = \frac{\text{parent function period}}{\mathrm{B}}.$$

Frequency

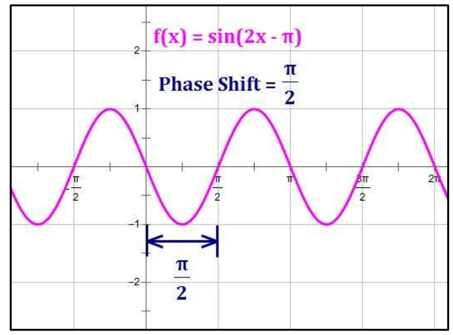
Frequency is most useful when used with the sine and cosine functions. It is the reciprocal of the period, i.e.,

$$\text{frequency} = \frac{1}{\text{period}}.$$

Frequency is typically discussed in relation to the sine and cosine functions when considering harmonic motion or waves. In Physics, frequency is typically measured in Hertz, i.e., cycles per second. 1 Hz = 1 cycle per second.

For the general sine or cosine function, $f(x)$, defined above, $\text{frequency} = \frac{\mathrm{B}}{2\pi}$.

Phase Shift



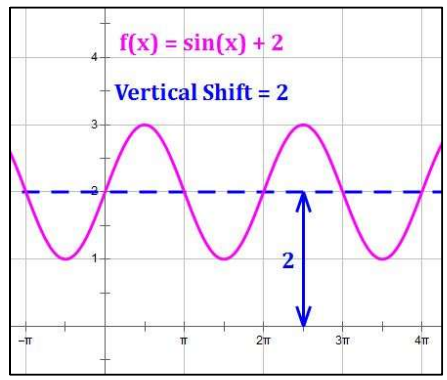
Phase shift is how far has the function been shifted horizontally (left or right) from its parent function. For the general function, $f(x)$, defined above,

$$\text{phase shift} = \frac{C}{B}.$$

A positive phase shift indicates a shift to the right relative to the graph of the parent function; a negative phase shift indicates a shift to the left relative to the graph of the parent function.

A trick for calculating the phase shift is to set the argument of the trigonometric function equal to zero: $(Bx - C) = 0$, and solve for x. The resulting value of x is the phase shift of the function.

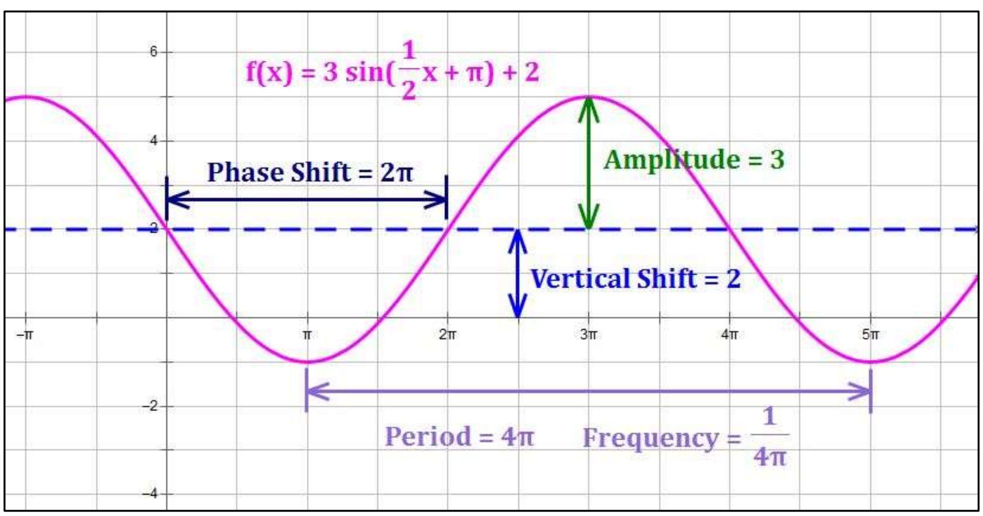
Vertical Shift



Vertical shift is the vertical distance that the midline of a curve lies above or below the midline of its parent function (i.e., the x-axis). For the general function, $f(x)$, defined above, $\text{vertical shift} = D.$ The value of D may be positive, indicating a shift upward, or negative, indicating a shift downward relative to the graph of the parent function.

Putting it All Together

The illustration below shows how all of the items described above combine in a single graph.



For 29 – 36, graph the function on the provided coordinate plain.

There are numerous methods that can be used to graph Trig functions. The one I will use here is to identify a window within which one period exists, to graph that one period, and finally to extend the graph left and right to complete the graph over the interval requested, $[-\pi, 2\pi]$ or $[-2\pi, 2\pi]$, as identified in each problem.

The student should use which ever graphing method they find most understandable. If you have a method that works better for you than what is shown here, you should use that method.

29) $y = 3\sin(2x)$

$y = 3\sin(2x)$

Step 1: Curve characteristics:

- Amplitude: 3
- Period: One period of $\sin x$ exists on the interval $[0, 2\pi]$. Divide by 2 to get the interval $[0, \pi]$.
- Phase Shift: There is no addition or subtraction in the parentheses, so there is no phase shift.
- Vertical Shift: There is no addition or subtraction after the sine term, so there is no vertical shift. This means that the midline of the graph will be the x-axis.

Step 2: Identify the window within which one period exists (using the information in Step 1):

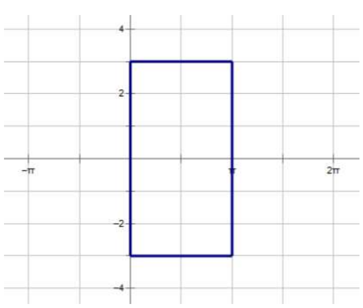
- Period: One period of our function exists on the interval $[0, \pi]$.
- Maximum value is 3, minimum value is -3.
- It may be helpful to draw a box showing this window.

Step 3: Graph one period of the function within the window in Step 2.

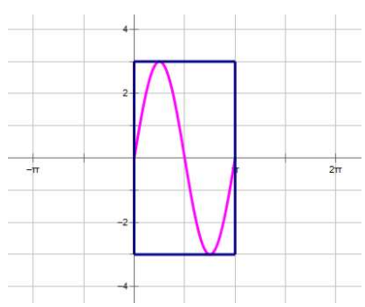
- $\sin x = 0$ at the beginning, middle and end of a period.
- $\sin x =$ its amplitude when x is half-way between the first two zeros.
- $\sin x =$ the negative of its amplitude when x is half-way between the second and third zeros.

Step 4: Repeat the graph over the required interval: $[-\pi, 2\pi]$.

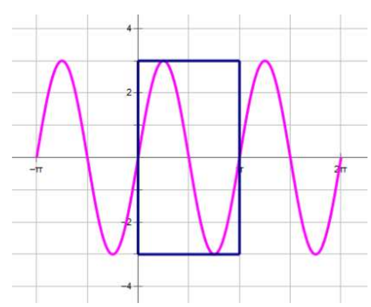
- You may choose to remove the window in your final graph, depending on your teacher's preference.



Step 2



Step 3



Step 4

30) $y = -2\sin\left(x + \frac{\pi}{4}\right) - 1$

$y = -2\sin\left(x + \frac{\pi}{4}\right) - 1$

Step 1: Curve characteristics:

- Amplitude: 2. The negative in front of the 2 indicates the curve is flipped upside down.
- Period: One period of $\sin x$ exists on the interval $[0, 2\pi]$.
- Phase Shift: left $\frac{\pi}{4}$ because of the $+$ sign in front of $\frac{\pi}{4}$.
- Vertical Shift: The entire curve is shifted down 1 unit, meaning the midline of the curve is no longer the x-axis. The new midline is $y = -1$.

Step 2: Identify the window within which one period exists (using the information in Step 1):

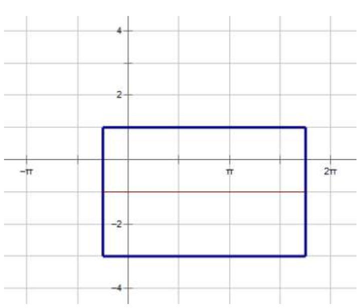
- Period: One period of our function exists on the interval $[0, 2\pi] - \frac{\pi}{4} = \left[-\frac{\pi}{4}, \frac{7\pi}{4}\right]$.
- Maximum value is $2 - 1 = 1$, minimum value is $-2 - 1 = -3$.
- It may be helpful to draw a box showing this window.

Step 3: Graph one period of the function within the window in Step 2.

- $\sin x$ crosses the midline at the beginning, middle, and end of a period.
- $\sin x =$ its maximum value when x is half-way between the first two crossings.
- $\sin x =$ its minimum value when x is half-way between the second and third crossings.
- The new midline is shown in brown. You do not need to graph it; it is only a guide for you.

Step 4: Repeat the graph over the required interval: $[-\pi, 2\pi]$.

- You may choose to remove the window and the midline in your final graph, depending on your teacher's preference.



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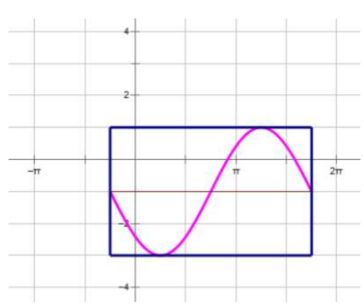
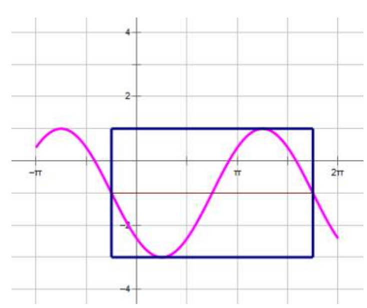


Step 3



Step 4

31) $y = 3\cos\left(\frac{1}{2}x\right)$

$y = 3\cos\left(\frac{1}{2}x\right)$

Step 1: Curve characteristics:

- Amplitude: 3
- Period: One period of $\cos x$ exists on the interval $[0, 2\pi]$. Divide by $\frac{1}{2}$ to get the interval $[0, 4\pi]$.
- Phase Shift: There is no addition or subtraction in the parentheses, so there is no phase shift.
- Vertical Shift: There is no addition or subtraction after the cosine term, so there is no vertical shift. This means that the midline of the graph will be the x-axis.

Step 2: Identify the window within which one period exists (using the information in Step 1):

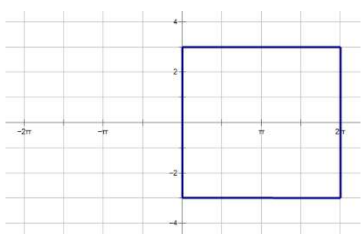
- Period: One period of our function exists on the interval $[0, 4\pi]$.
- Since the period is $[0, 4\pi]$, and we are required to draw on $[-2\pi, 2\pi]$, we will start with a window of $[0, 2\pi]$ and draw only the left half of the required function in this step.
- Maximum value is 3, minimum value is -3.
- It may be helpful to draw a box showing this window.

Step 3: Graph the left half of one period of the function within the window in Step 2.

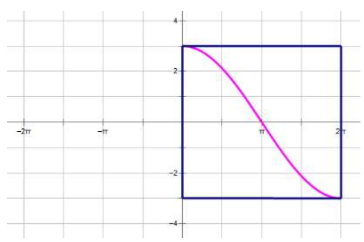
- $\cos x =$ its maximum value (amplitude) at the beginning and end of a period. Note that we have not included the end of the period in our window.
- $\cos x =$ its minimum value (negative amplitude) when x is half-way between the beginning and end of a period. Note that our window ends at this half-way point.
- $\cos x = 0$ half-way between its minimum and maximum values.

Step 4: Repeat the graph over the required interval: $[-2\pi, 2\pi]$.

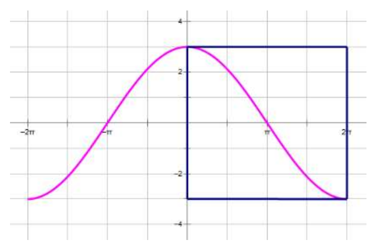
- You may choose to remove the window in your final graph, depending on your teacher's preference.



Step 2



Step 3



Step 4

32) $y = -3\cos(2x - \pi)$

$y = -3\cos(2x - \pi)$

Step 1: Curve characteristics:

- Amplitude: 3. The negative in front of the 3 indicates the curve is flipped upside down.
- Period: One period of $\cos x$ exists on the interval $[0, 2\pi]$. Divide by 2 to get the interval $[0, \pi]$.
- Phase Shift: To determine the amount of shift, set $2x - \pi = 0$ and solve for x. $x = \frac{\pi}{2}$, indicating a shift right of $\frac{\pi}{2}$.
- Vertical Shift: There is no addition or subtraction after the cosine term, so there is no vertical shift. This means that the midline of the graph will be the x-axis.

Step 2: Identify the window within which one period exists (using the information in Step 1):

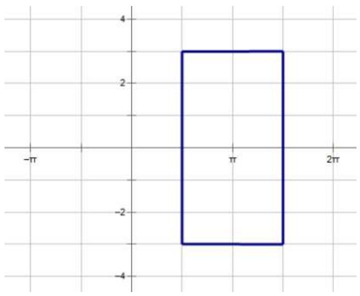
- Period: One period of our function exists on the interval $[0, \pi] + \frac{\pi}{2} = \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.
- Maximum value is 3, minimum value is -3.
- It may be helpful to draw a box showing this window.

Step 3: Graph one period of the function within the window in Step 2.

- $\cos x =$ its maximum value (amplitude) at the beginning and end of a period. Since our curve is inverted, we will graph the minimum values at the beginning and end of our period.
- $\cos x =$ its minimum value (negative amplitude) when x is half-way between the beginning and end of a period. Since our curve is inverted, we will graph the maximum value at the half-way point.
- $\cos x = 0$ half-way between its minimum and maximum values.

Step 4: Repeat the graph over the required interval: $[-\pi, 2\pi]$.

- You may choose to remove the window and the midline in your final graph, depending on your teacher's preference.



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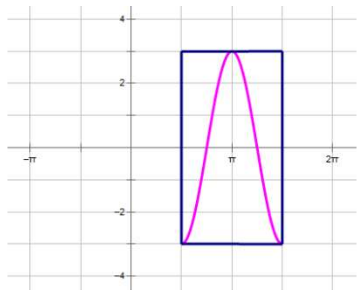

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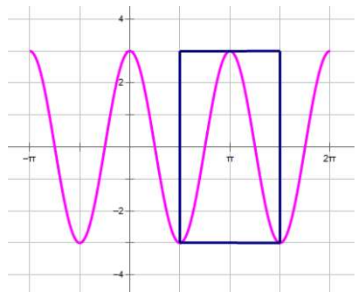


Step 4

33) $y = \tan(2x)$

$y = \tan(2x)$

Step 1: Curve characteristics:

- Amplitude: 1
- Period: One period of $\tan x$ exists on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Divide by 2 to get the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$.
- Phase Shift: There is no addition or subtraction in the parentheses, so there is no phase shift.
- Vertical Shift: There is no addition or subtraction after the tangent term, so there is no vertical shift. This means that the midline of the graph will be the x-axis.

Step 2: Identify the window within which one period exists (using the information in Step 1):

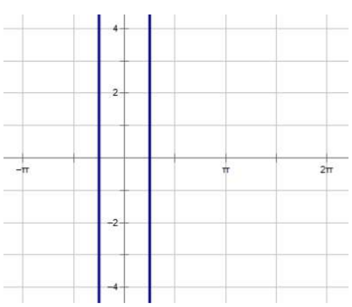
- Period: One period of our function exists on the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$.
- The tangent function does not have maximums or minimums.
- The window for tangent may best be shown as the area between vertical lines (asymptotes) at the endpoints of our period.

Step 3: Graph one period of the function within the window in Step 2.

- $\tan x = 0$ at the middle a period.
- $\tan x =$ its amplitude when x is half-way between the middle and end of the period. This point is shown on the graph. This is not a maximum; it is a graphing guide point.
- $\tan x =$ the negative of its amplitude when x is half-way between the beginning and middle of the period. This point is shown on the graph. This is not a minimum; it is a graphing guide point.
- $\tan x$ starts on the left of its period at $-\infty$ and proceeds upward toward ∞ as it approaches the end of its period.

Step 4: Repeat the graph over the required interval: $[-\pi, 2\pi]$.

- You may choose to remove the window in your final graph, depending on your teacher's preference.



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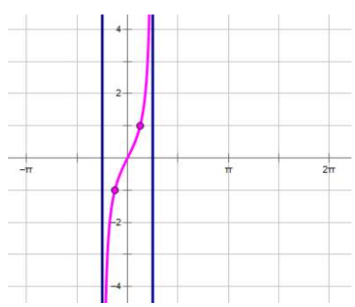

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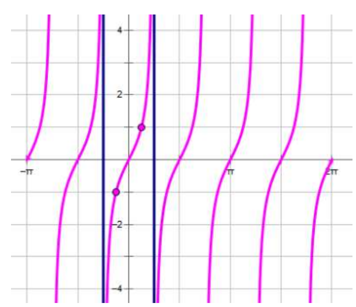


Step 4

34) $y = 4\cot\left(\frac{x}{2}\right)$

$y = 4\cot\left(\frac{1}{2}x\right)$

Step 1: Curve characteristics:

- Amplitude: 4
- Period: One period of $\cot x$ exists on the interval $[0, \pi]$. Divide by $\frac{1}{2}$ to get the interval $[0, 2\pi]$.
- Phase Shift: There is no addition or subtraction in the parentheses, so there is no phase shift.
- Vertical Shift: There is no addition or subtraction after the cotangent term, so there is no vertical shift. This means that the midline of the graph will be the x-axis.

Step 2: Identify the window within which one period exists (using the information in Step 1):

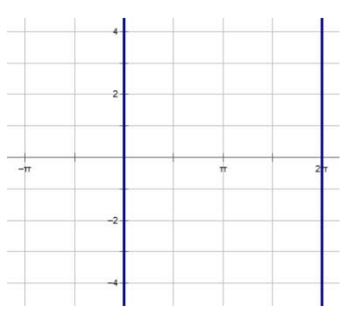
- Period: One period of our function exists on the interval $[0, 2\pi]$.
- The cotangent function does not have maximums or minimums.
- The window for cotangent may best be shown as the area between vertical lines (asymptotes) at the endpoints of our period.

Step 3: Graph one period of the function within the window in Step 2.

- $\cot x = 0$ at the middle a period.
- $\cot x =$ its amplitude when x is half-way between the beginning and middle of the period. This point is shown on the graph. This is not a maximum; it is a graphing guide point.
- $\cot x =$ the negative of its amplitude when x is half-way between the middle and end of the period. This point is shown on the graph. This is not a minimum; it is a graphing guide point.
- $\cot x$ starts on the left of its period at ∞ and proceeds downward toward $-\infty$ as it approaches the end of its period.

Step 4: Repeat the graph over the required interval: $[-\pi, 2\pi]$.

- You may choose to remove the window in your final graph, depending on your teacher's preference.



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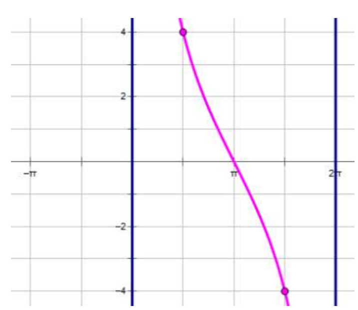

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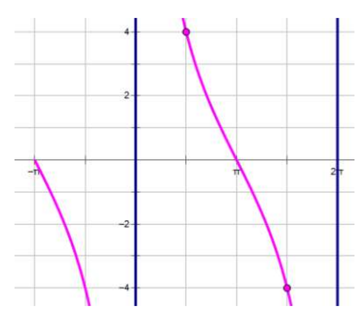


Step 4

35) $y = -\sec\frac{1}{2}x$

$$y = -\sec\left(\frac{1}{2}x\right) = -\frac{1}{\cos\left(\frac{1}{2}x\right)}$$

Step 1: Curve characteristics:

- Amplitude: 1. The negative in front of the sec indicates the curve is flipped upside down.
- We will use the corresponding cosine function to help graph the secant function.
- Period: One period of $\cos x$ exists on the interval $[0, 2\pi]$. Divide by $\frac{1}{2}$ to get the interval $[0, 4\pi]$.
- Phase Shift: There is no addition or subtraction in the parentheses, so there is no phase shift.
- Vertical Shift: There is no addition or subtraction after the secant term, so there is no vertical shift. This means that the midline of the graph will be the x-axis.

Step 2: Identify the window within which one period exists (using the information in Step 1):

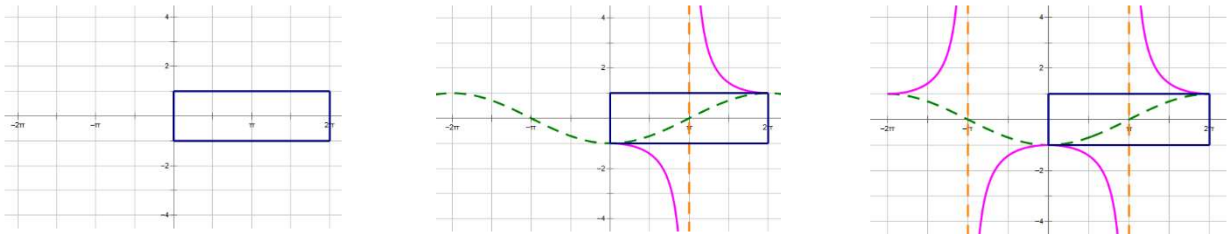
- Period: One period of our function exists on the interval $[0, 4\pi]$.
- Since the period is $[0, 4\pi]$, and we are required to draw on $[-2\pi, 2\pi]$, we will start with a window of $[0, 2\pi]$ and draw only the left half of the required functions.
- Maximum cosine value is 1, minimum cosine value is -1.
- It may be helpful to draw a box showing this window.

Step 3: Graph the left half of one period of the cosine and secant functions within the window in Step 2.

- $\cos x =$ its maximum value (amplitude) at the beginning and end of a period. Note that we have not included the end of the period in our window. Since our curve is inverted, we will graph the minimum values at the beginning of our period.
- $\cos x =$ its minimum value (negative amplitude) when x is half-way between the beginning and end of a period. Note that our window ends at this half-way point. Since our curve is inverted, we will graph the maximum value at this half-way point.
- $\cos x = 0$ half-way between its minimum and maximum values.
- The secant function has asymptotes where cosine is zero and inverts the cosine function elsewhere.
- The secant and cosine functions touch at the cosine function's highest and lowest points.

Step 4: Repeat the graph over the required interval: $[-2\pi, 2\pi]$.

- You may choose to remove the window in your final graph, depending on your teacher's preference.
- The violet line is the desired secant function.



Step 2 Step 3 Step 4

36) $y = 2\csc(x + \frac{\pi}{2})$

$$y = 2\csc\left(x + \frac{\pi}{2}\right) = \frac{1}{\sin\left(x + \frac{\pi}{2}\right)}$$

Step 1: Curve characteristics:

- Amplitude: 2.
- We will use the corresponding sine function to help graph the cosecant function.
- Period: One period of $\sin x$ exists on the interval $[0, 2\pi]$.
- Phase Shift: left $\frac{\pi}{2}$ because of the $+$ sign in front of $\frac{\pi}{2}$.
- Vertical Shift: There is no addition or subtraction after the cosecant term, so there is no vertical shift. This means that the midline of the graph will be the x-axis.

Step 2: Identify the window within which one period exists (using the information in Step 1):

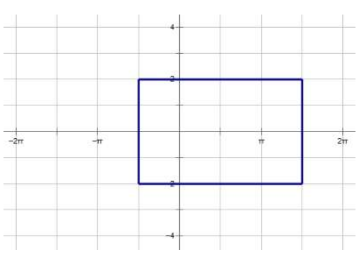
- Period: One period of our function exists on the interval $[0, 2\pi] - \frac{\pi}{2} = \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.
- Maximum sine value is 2, minimum sine value is -2.
- It may be helpful to draw a box showing this window.

Step 3: Graph one period of the function within the window in Step 2.

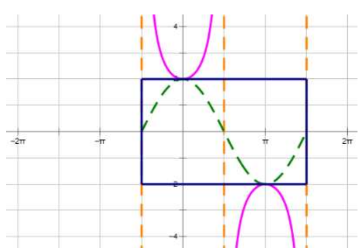
- $\sin x = 0$ at the beginning, middle, and end of a period.
- $\sin x =$ its maximum value (amplitude) when x is half-way between its first two zeros in a period.
- $\sin x =$ its minimum value (negative amplitude) when x is half-way between the second and third zeros in a period.
- The cosecant function has asymptotes where sine is zero and inverts the sine function elsewhere.
- The cosecant and sine functions touch at the sine function's highest and lowest points.

Step 4: Repeat the graph over the required interval: $[-2\pi, 2\pi]$.

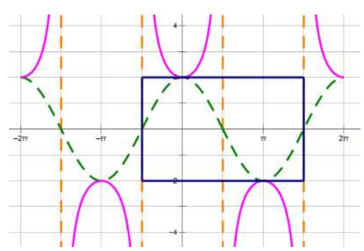
- You may choose to remove the window in your final graph, depending on your teacher's preference.
- The violet line is the desired cosecant function.



Step 2



Step 3



Step 4

37) Describe the transformations and period for $y = -4\cos(3x + 1) - 3$.

$y = -4\cos(3x + 1) - 3.$

Transformations:

- Negative sign: **reflection over the x-axis.**
- 4 is a **stretch in the y-direction by a factor of 4.**
- 3 is a **compression in the x-direction by a factor of 3** (3 periods in the space of one period in the parent function). The sized of the period for the parent function (cosine) is 2π, and **the period for this function is $\frac{2\pi}{3}$.**
- Phase Shift: To determine the amount of shift, set $(3x + 1) = 0$ and solve for x. $x = -\frac{1}{3}$, indicating a **shift left of $\frac{1}{3}$ radian.**
- -3 indicates a **shift of the entire curve downward by 3 units** after the reflection over the x-axis.

For 38 – 41, find the exact value of the expression, use the restricted range.

38) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

This is the angle θ in either quadrant 1 or 4 for which $\sin\theta = -\frac{\sqrt{3}}{2}$. $\boldsymbol{\theta = -\frac{\pi}{3}}$.

39) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

This is the angle θ in either quadrant 1 or 2 for which $\cos\theta = -\frac{\sqrt{2}}{2}$. $\boldsymbol{\theta = \frac{3\pi}{4}}$.

40) $\cos^{-1}(1)$

This is the angle θ for which $\cos\theta = 1$. $\boldsymbol{\theta = 0}$.

41) $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

This is the angle θ in either quadrant 1 or 4 for which $\tan\theta = \frac{\sqrt{3}}{3}$. $\boldsymbol{\theta = \frac{\pi}{6}}$.

For 42 – 47, find the value of the expression. Use a sketch but do NOT use a calculator.

Inverse tangent functions are each defined in two quadrants. Sine, tangent, and secant are defined in quadrants 1 and 4. Cosine and cosecant are defined in quadrants 1 and 2. Cotangent is an odd bird for which the quadrants are in dispute among mathematicians. Fortunately, the solution to Problem 43 is in quadrant 1, which is not is dispute.

42) $\tan^{-1}[\tan\left(\frac{\pi}{5}\right)]$

Since the angle $\frac{\pi}{5}$ is in quadrant 1 and tan and $\tan^{-1}$ are inverses, the solution is the angle $\frac{\pi}{5}$.

43) $\cot[\sin^{-1}\left(\frac{5}{7}\right)]$

Since $\sin^{-1}\left(\frac{5}{7}\right)$ is in quadrant 1 (because $\frac{5}{7}$ is positive), we can solve this problem without dispute (see the comment above about the inverse cotangent function). The cotangent function in quadrant 1 is also positive. We could draw a triangle, but let's have a little more fun with it.

We want to know the cotangent of the angle θ that has a sine value of $\frac{5}{7}$. Note that $\sin\theta = \frac{5}{7}$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\frac{5}{7}\right)^2 + \cos^2\theta = 1$$

$$\frac{25}{49} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{25}{49} = \frac{24}{49}$$

$$\cos\theta = \sqrt{\frac{24}{49}} = \frac{2\sqrt{6}}{7}$$

Then,

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{2\sqrt{6}/7}{5/7} = \mathbf{\frac{2\sqrt{6}}{5}}$$

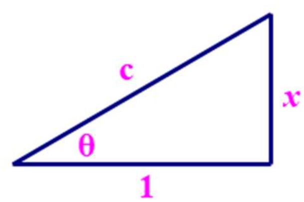
44) $\cos(\tan^{-1}x);\ x > 0$

Let's draw this one. We need an angle that has a tangent value of x.

$$c^2 = x^2 + 1^2 \quad \Rightarrow \quad c = \sqrt{x^2+1}$$

$$\theta = \tan^{-1}x$$

$$\cos\theta = \cos(\tan^{-1}x) = \frac{1}{c} = \frac{1}{\sqrt{x^2+1}} = \mathbf{\frac{\sqrt{x^2+1}}{x^2+1}}$$



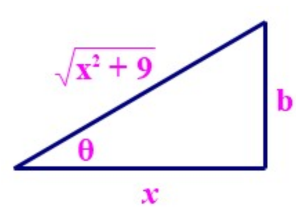
45) $\sin[\sec^{-1}\left(\frac{\sqrt{x^2+9}}{x}\right)];\ x > 0$

Let's draw this one. We need an angle that has a secant value of $\frac{\sqrt{x^2+9}}{x}$. In more comfortable terms, the angle must have a cosine value of $\frac{x}{\sqrt{x^2+9}}$.

$$x^2 + b^2 = \left(\sqrt{x^2+9}\right)^2 \quad \Rightarrow \quad b = 3$$

$$\theta = \sec^{-1}\left(\frac{\sqrt{x^2+9}}{x}\right)$$

$$\sin\theta = \sin\left[\sec^{-1}\left(\frac{\sqrt{x^2+9}}{x}\right)\right] = \frac{b}{\sqrt{x^2+9}} = \frac{3}{\sqrt{x^2+9}} = \mathbf{\frac{3\sqrt{x^2+9}}{x^2+9}}$$



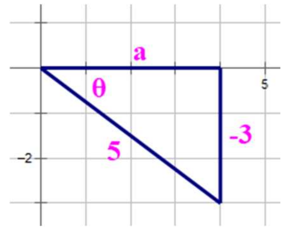
46) $\cos[\sin^{-1}\left(-\frac{3}{5}\right)]$

Let's draw this one on a coordinate system. The angle in question must have a sine value of $-\frac{3}{5}$ and must reside in Q4, since that is where the inverse sine function is defined when the sine value is negative.

$a^2 + (-3)^2 = 5^2 \quad \Rightarrow \quad a = 4$

$\theta = \sin^{-1}\left(-\frac{3}{5}\right)$

$\cos\theta = \cos\left[\sin^{-1}\left(-\frac{3}{5}\right)\right] = \frac{a}{5} = \frac{\mathbf{4}}{\mathbf{5}}$



47) $\sin^{-1}[\sin\left(\frac{13\pi}{7}\right)]$

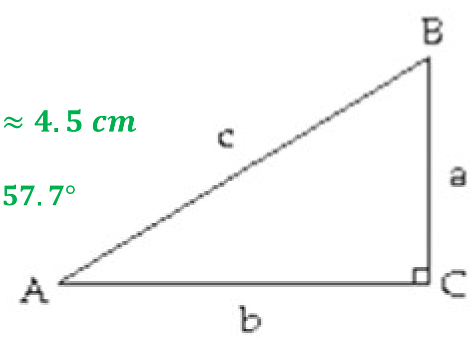
Let's reason this one out (no drawing) because sine and inverse sine are inverse functions. The angle $\frac{13\pi}{7}$ is in Q4, but is not in the interval required for inverse sine: $[-\pi, \pi]$.

The angle $\frac{13\pi}{7}$ is the same as the angle $\frac{13\pi}{7} - 2\pi = -\frac{\pi}{7}$ due to the periodic nature of the sine function. The angle $-\frac{\pi}{7}$ is within the interval required for inverse sine: $[-\pi, \pi]$. So,

$$\sin^{-1}\left[\sin\left(\frac{13\pi}{7}\right)\right] = \sin^{-1}\left[\sin\left(-\frac{\pi}{7}\right)\right] = -\frac{\boldsymbol{\pi}}{\mathbf{7}}$$

***For 48 – 49, solve the right triangle shown in the figure. Round all answers to the nearest tenth.**

Hello, Pythagoras!



48) $a = 3.8\ cm, b = 2.4\ cm$

$c^2 = a^2 + b^2 = 3.8^2 + 2.4^2 = 20.2 \quad \Rightarrow \quad \boldsymbol{c} = \sqrt{20.2} \approx 4.49444 \approx \mathbf{4.5\ cm}$

$\sin A = \frac{a}{c} = \frac{3.8}{4.49444} = 0.84549 \quad \Rightarrow \quad \boldsymbol{m\angle A} = \sin^{-1}(0.84549) = \mathbf{57.7°}$

$\boldsymbol{m\angle B} = 90° - 57.7° = \mathbf{32.3°}$

49) $a = 3.3\ in, A = 55.1°$

$\sin A = \frac{a}{c} \quad \Rightarrow \quad \sin(55.1°) = \frac{3.3}{c} \quad \Rightarrow \quad \boldsymbol{c} = \frac{3.3}{\sin(55.1°)} \approx 4.023645 \approx \mathbf{4.0\ in}$

$c^2 = a^2 + b^2 \quad \Rightarrow \quad 4.023645^2 = 3.3^2 + b^2 \quad \Rightarrow \quad \boldsymbol{b} = \sqrt{4.023645^2 - 3.3^2} \approx 2.30211 \approx \mathbf{2.3\ in}$

$\boldsymbol{m\angle B} = 90° - 55.1° = \mathbf{34.9°}$

For 50 – 51, convert the following coordinates from either polar to rectangular or rectangular to polar.

50) $(-3, 120°)$ The point is in polar form. Convert to rectangular form.

The point at these polar coordinates is a distance 3 from the origin and at an angle of $120° + 180° = 300°$. Note that we add $180°$ because the $r = -3$ is negative.

Then,

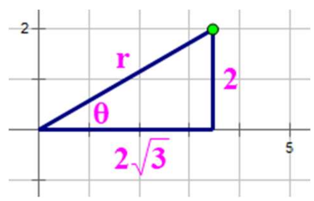
$$x = r\cos\theta = -3 \cdot \cos 120° = 3 \cdot \cos 300° = 3 \cdot \left(\frac{1}{2}\right) = \frac{3}{2}$$

$$y = r\sin\theta = -3 \cdot \sin 120° = 3 \cdot \sin 300° = 3 \cdot \left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2}$$

So, the point's rectangular coordinates are: $\left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$

51) $(2\sqrt{3}, 2)$ (answer in radians) The point is in rectangular (Cartesian) form. Convert to polar form.

Let's draw this point:



$x = 2\sqrt{3}$

$y = 2$

$r = \sqrt{\left(2\sqrt{3}\right)^2 + 2^2} = 4$

$\sin\theta = \frac{2}{r} = \frac{2}{4} = \frac{1}{2} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

So, the point's polar coordinates are: $\left(4, \frac{\pi}{6}\right)$

For 52 – 53, convert the following equations from either polar to rectangular or rectangular to polar.

52) $(x-16)^2 + y^2 = 256$

Note that this is a circle of radius $\sqrt{256} = 16$

Recall: $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$

$(x-16)^2 + y^2 = 256$

$(x^2 - 32x + 256) + y^2 = 256$

$x^2 + y^2 - 32x = 0$

$r^2 - 32r\cos\theta = 0$

$r(r - 32\cos\theta) = 0$

$r = 0$ or $r = 32\cos\theta$

$r = 0$ is not part of the circle defined above, so discard it.

So, $\boldsymbol{r = 32\cos\theta}$

53) $r = 4\csc\theta$

$r = 4\csc\theta$

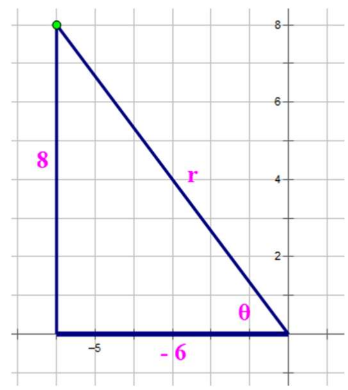
$r = \dfrac{4}{\sin\theta}$

$r\sin\theta = 4$

$\boldsymbol{y = 4}$ because $y = r\sin\theta$

*54) Write the complex number, $-6 + 8i$, in polar form.

This very similar to converting a point to polar form. Let's examine this number, which lies in Q2:



$x = -6$

$y = 8$

$r = \sqrt{x^2 + y^2} = \sqrt{(-6)^2 + 8^2} = 10$

$\sin\theta = \dfrac{8}{r} = \dfrac{8}{10} = \dfrac{4}{5} \quad \Rightarrow \quad \sin^{-1}\left(\dfrac{4}{5}\right) = 0.927\,rad = 53.1°$

θ cannot be $53.1°$ because θ is in Q2. θ, then is the angle in Q2 that has the same sine value as $53.1°$; θ is supplementary to $53.1°$.

$\theta = 180° - 53.1° = 126.9°$

So, the point's polar form is: $\mathbf{10\ cis\ 126.9°}$

55) Write the complex number, $3(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$, in rectangular form.

We can simply expand this complex number to get it's complex (rectangular) form:

$3\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

$= 3\left(-\dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}\right)$

$= \boldsymbol{-\dfrac{3\sqrt{2}}{2} + \dfrac{3\sqrt{2}}{2}i}$

56) Find the product of the complex numbers.

$z_1 = 5(\cos 200° + i\sin 200°)$

$z_2 = 4(\cos 50° + i\sin 50°)$

57) Find the quotient of the complex numbers.

$z_1 = 5(\cos 20° + i\sin 20°)$

$z_2 = 4(\cos 10° + i\sin 10°)$

Using DeMoivre's Theorem for multiplication or division, scalars are multiplied or divided and angles are added or subtracted.

$5(\cos 200° + i\sin 200°)$
$\cdot\ 4(\cos 50° + i\sin 50°)$

$\mathbf{20\ cis\ 250°}$

$5(\cos 20° + i\sin 20°)$
$\div\ 4(\cos 10° + i\sin 10°)$

$\mathbf{\frac{5}{4}\ cis\ 10°}$

58) Use DeMoivre's Theorem to find the indicated power of the complex number. Write the answer in polar (cis) form. $\left[10\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^3$

Using DeMoivre's Theorem for powers, scalars are taken to powers and angles are multiplied by exponents.

$\left[10\left(\text{cis}\frac{3\pi}{4}\right)\right]^3 = 10^3\text{cis}\left(3\cdot\frac{3\pi}{4}\right) = 1000\text{ cis}\left(\frac{9\pi}{4}\right) = 1000\text{ cis}\left(\frac{\pi}{4}\right)$ (by subtracting 2π from $\frac{9\pi}{4}$)

59) Given $\mathbf{u} = -12\mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = 6\mathbf{i} - 7\mathbf{j}$; Find $3\mathbf{u} - 6\mathbf{v}$.

$$\begin{array}{lll} -12\mathbf{i} - 2\mathbf{j} & \cdot(3) = & -36\mathbf{i} - 6\mathbf{j} \\ 6\mathbf{i} - 7\mathbf{j} & \cdot(-6) = & -36\mathbf{i} + 42\mathbf{j} \\ \hline & & \mathbf{-72i + 36j} \end{array}$$

*60) Given $\vec{v} = 10\mathbf{i} - 4\mathbf{j}$; Find $\|-7\vec{v}\|$. Round answer to the nearest tenth.

$\vec{v} = 10\mathbf{i} - 4\mathbf{j}$

$\|\vec{v}\| = \sqrt{10^2 + (-4)^2} = \sqrt{116} = 2\sqrt{29}$

$\|-7\vec{v}\| = |-7|\cdot\|\vec{v}\| = 7\cdot 2\sqrt{29} = 14\sqrt{29} \approx \mathbf{75.4}$

*61) Given that $\vec{v} = -3\mathbf{i} - 4\mathbf{j}$ and $\vec{w} = 6\mathbf{i} + 8\mathbf{j}$. Find the measure of the angle (in degrees) between the vectors, and classify the vectors as parallel, orthogonal, or neither.

Just looking at the vectors, we can see that $\vec{w} = -2\cdot\vec{v}$. Since one vector is a multiple of the other, the vectors are parallel.

The negative sign in this relationship means the vectors are heading in opposite directions, so the angle between them is 180°. Vectors with a 180° between them are often called antiparallel to distinguish them from parallel vectors with an angle of 0° between them.

Now that we have solved the problem by using vector properties, let's do it with some Trigonometry.

$$\cos\theta = \frac{\vec{v}\circ\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|}$$

$\vec{v}\circ\vec{w} = (-3)\cdot 6 + (-4)\cdot 8 = -50$

$\|\vec{v}\| = \sqrt{(-3)^2 + (-4)^2} = 5$

$\|\vec{w}\| = \sqrt{6^2 + 8^2} = 10$

$$\cos\theta = \frac{\vec{v}\circ\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|} = \frac{-50}{5\cdot 10} = -1 \quad\Rightarrow\quad \boldsymbol{\theta = 180°}$$

If the angle between two vectors is 0° or 180°, they are technically parallel, so **$\vec{v}$ and $\vec{w}$ are parallel**.

(See also the comment above about anti-parallel lines so you can be even smarter!)

62) Given $\vec{v}$ with $\|\vec{v}\| = 7$ and $\theta = 225°$. Write the vector $\vec{v}$ in terms of **i** and **j**.

$$x = \|\vec{v}\| \cos\theta = 7 \cdot \cos 225° = 7 \cdot \left(\frac{-\sqrt{2}}{2}\right) = -\frac{7\sqrt{2}}{2}$$

$$y = \|\vec{v}\| \sin\theta = 7 \cdot \sin 225° = 7 \cdot \left(\frac{-\sqrt{2}}{2}\right) = -\frac{7\sqrt{2}}{2}$$

So,

$$\vec{v} = -\frac{7\sqrt{2}}{2}\mathbf{i} - -\frac{7\sqrt{2}}{2}\mathbf{j}$$

*63) Two airplanes leave an airport at the same time on different runways. One flies S 15° E at 250 mph. The other airplane flies on a bearing of N 63° E at 302 mph. How far apart will the airplanes be after 2.5 hours? Round to the nearest mile.

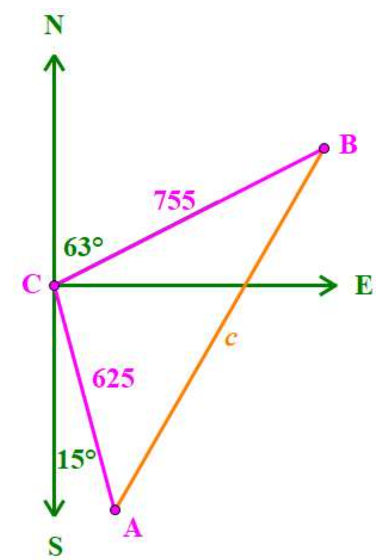
We can solve this problem with vectors or with the Law of Cosines. Let's use the Law of Cosines since that involves less work.

Plane A travels 250 $mph \cdot 2.5\ hours = 625\ miles$ at a bearing of S 15° E.

Plane B travels 302 $mph \cdot 2.5\ hours = 755\ miles$ at a bearing of N 63° E.

In the diagram to the right,

- The flight paths of the two planes are shown in magenta.
- The angle between the paths of the planes is $180° - 63° - 15° = 102°$.
- The distance between planes A and B is c.



Calculate the distance c using the Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = 625^2 + 755^2 - 2 \cdot 625 \cdot 755 \cdot \cos 102°$$
$$c^2 \approx 1{,}156{,}866.658$$
$$c \approx \mathbf{1{,}076\ miles}$$

64) Find the unit vector of the following vector, $\vec{v} = 3\mathbf{i} - 4\mathbf{j}$.

Unit vectors are often shown with a hat on them. Also, a unit vector has a magnitude of 1.

Let $\mathbf{v} = \vec{v}$. Then, the requested unit vector (shown with a hat) is:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3\mathbf{i} - 4\mathbf{j}}{\sqrt{3^2 + (-4)^2}} = \frac{3\mathbf{i} - 4\mathbf{j}}{\mathbf{5}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

65) Write an equivalent representation in the form (r, θ) for the point $\left(2, \frac{5\pi}{6}\right)$, where $r < 0$ and $-2\pi < \theta < 0$?

Adding π to, or subtracting π from, and angle changes the sign of the r-value (2 in this problem).

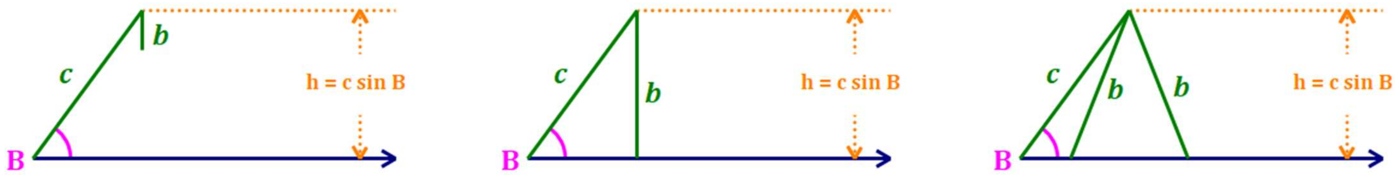
Subtracting π from the angle given will generate an angle in the required interval: $\frac{5\pi}{6} - \pi = -\frac{\pi}{6}$. This requires us to change the sign of the r-value from 2 to -2, which meets the requirement for r. No further manipulation is required.

The equivalent representation is: $\left(\mathbf{-2}, -\frac{\pi}{6}\right)$

*66) Identify the number of possible triangles. Then solve for all missing sides and angles. Round all values to the nearest tenth. b = 15, c = 17, B = 42°

I believe students at DRHS are taught a different method from the one I use, so you may choose to skip this problem and solve it your own way. But the method below is my way:

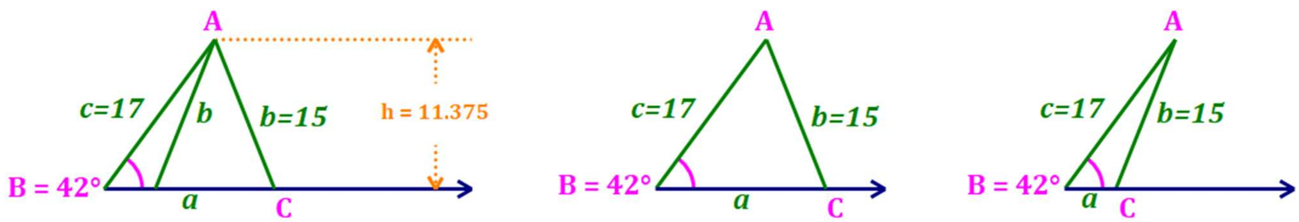
This is a Side-Side-Angle (SSA) problem with known angle B. There are 3 possible situations that arise, depending on how b compares to $h = c \cdot \sin B$, which is why this is called the Ambiguous Case:



So, let's begin with the comparison of b to $h = c \cdot \sin B$.

$h = c \cdot \sin B = 17 \cdot \sin 42° = 11.375 < b$ which is given as 15.

Therefore, we have the third case above, which is **2 triangles**. To redraw this case, with the given values:



Ambiguous Case breaks into: **Case 1** **Case 2**

The two cases will have the same value of $\sin C$, which is associated with two supplementary angles, one acute angle and one obtuse angle, giving rise to two possible triangles.

Using the Law of Sines,

$$\frac{\sin C}{c} = \frac{\sin B}{b} \quad \Rightarrow \quad \frac{\sin C}{17} = \frac{\sin 42°}{15} \quad \Rightarrow \quad \sin C = 0.75835$$

To find $m\angle C$: $\sin^{-1} 0.75835 \approx 49.3°$ for Case 1. We also want $180° - 49.3° = 130.7°$ for Case 2.

Case 1

$m\angle B = 42°$ $\quad b = 15$

$m\angle C = 49.3°$ $\quad c = 17$

$$m\angle A = 180° - 42° - 49.3° = 88.7°$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \Rightarrow \quad \frac{a}{\sin 88.7°} = \frac{15}{\sin 42°}$$

$$a = \frac{15 \ \sin 88.7°}{\sin 42°} = 22.4$$

$m\angle A = 88.7°$ $\quad a = 22.4$

Case 2

$m\angle B = 42°$ $\quad b = 15$

$m\angle C = 130.7°$ $\quad c = 17$

$$m\angle A = 180° - 42° - 130.7° = 7.3°$$

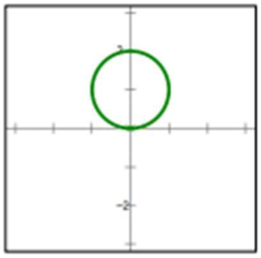
$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \Rightarrow \quad \frac{a}{\sin 7.3°} = \frac{15}{\sin 42°}$$

$$a = \frac{15 \ \sin 7.3°}{\sin 42°} = 2.8$$

$m\angle A = 7.3°$ $\quad a = 2.8$

Graph of Polar Equations

Circle

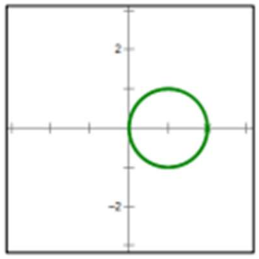


Equation: $r = a\sin\theta$

Location:
above x-axis if $a > 0$
below x-axis if $a < 0$

Radius: $a/2$

Symmetry: y-axis

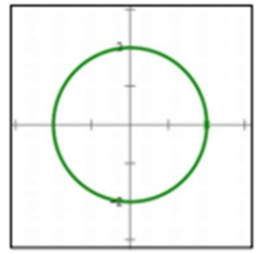


Equation: $r = a\cos\theta$

Location:
right of y-axis if $a > 0$
left of y-axis if $a < 0$

Radius: $a/2$

Symmetry: x-axis



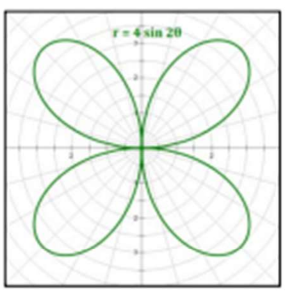
Equation: $r = a$

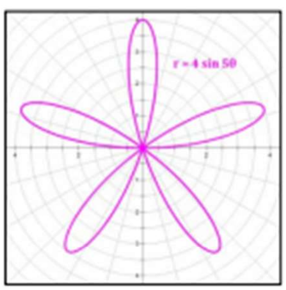
Location:
Centered on the Pole

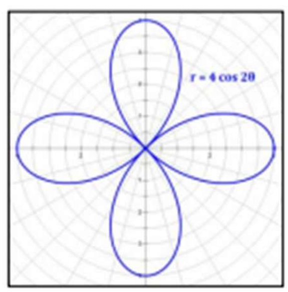
Radius: a

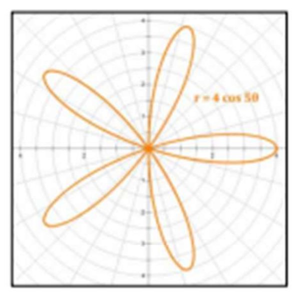
Symmetry: Pole, x-axis, y-axis

Rose







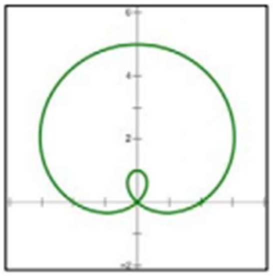


Characteristics of roses:

- Equation: $r = a\sin n\theta$
  - Symmetric about the y-axis
- Equation: $r = a\cos n\theta$
  - Symmetric about the x-axis
- Contained within a circle of radius $r = a$
- If n is odd, the rose has n petals.
- If n is even the rose has $2n$ petals.
- Note that a circle is a rose with one petal (i.e, $n = 1$).

Graphs of Polar Equations

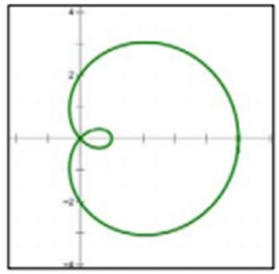
Limaçon of Pascal



Equation: $r = a + b\sin\theta$

Location: bulb above x-axis if $b > 0$
bulb below x-axis if $b < 0$

Symmetry: y-axis

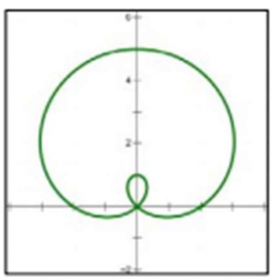


Equation: $r = a + b\cos\theta$

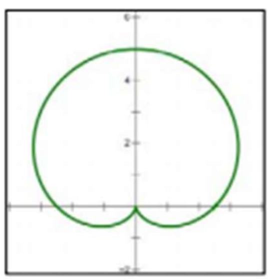
Location: bulb right of y-axis if $b > 0$
bulb left of y-axis if $b < 0$

Symmetry: x-axis

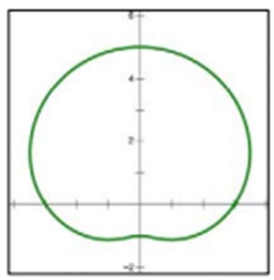
Four Limaçon Shapes



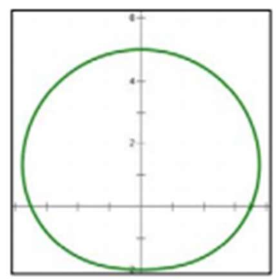
$a < b$
Inner loop



$a = b$
"Cardioid"

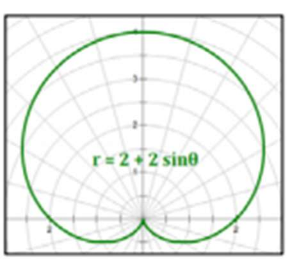


$b < a < 2b$
Dimple

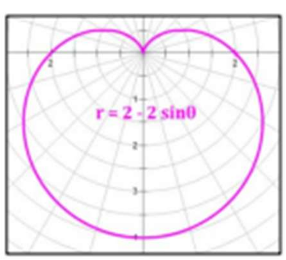


$a \geq 2b$
No dimple

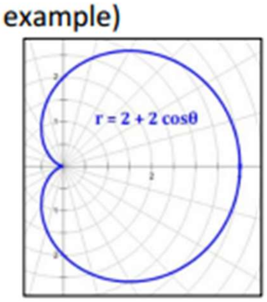
Four Limaçon Orientations (using the Cardioid as an example)



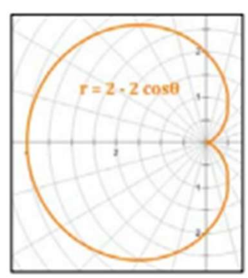
sine function
$b > 0$



sine function
$b < 0$



cosine function
$b > 0$



cosine function
$b < 0$

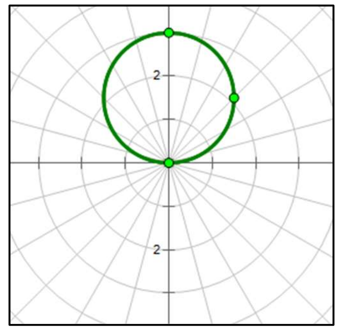
For 67- 71, graph each polar curve. An optional table is available for you to use, if desired.

Points in the tables are shown on the curves in light green.

67) Graph $r = 3\sin\theta$.

This is a graph of a circle that is symmetric about the y-axis (because the equation includes the sine function) and above the x-axis (because 3 is positive).

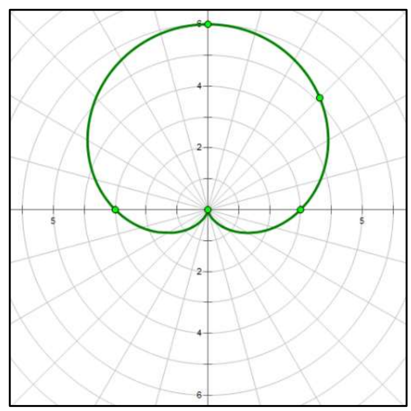
θ	$r = 3\sin\theta$
0	$r = 3\sin(0) = 0$
$\pi/4$	$r = 3\sin(\pi/4) \approx 2.12$
$\pi/2$	$r = 3\sin(\pi/2) = 3$
π	$r = 3\sin(\pi) = 0$
$3\pi/2$	$r = 3\sin(3\pi/2) = -3$



68) Graph $r = 3 + 3\sin\theta$.

This is a graph of a cardioid (a type of limaçon) because the constant and the coefficient of sine are equal. Cardioids have a cusp (pointy structure) at the origin. This cardioid is symmetric about the y-axis and mostly above the x-axis.

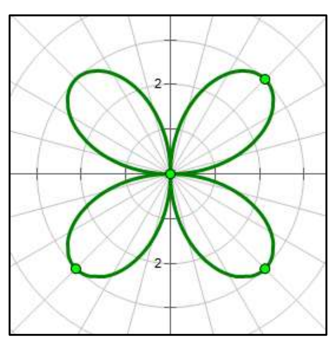
θ	$r = 3 + 3\sin\theta$
0	$r = 3 + 3\sin(0) = 3$
$\pi/4$	$r = 3 + 3\sin(\pi/4) \approx 5.12$
$\pi/2$	$r = 3 + 3\sin(\pi/2) = 6$
π	$r = 3 + 3\sin(\pi) = 3$
$3\pi/2$	$r = 3 + 3\sin(3\pi/2) = 0$



69) Graph $r = 3\sin 2\theta$.

This is a graph of a rose with four petals, one in each quadrant. Even-angle coefficients double the number of petals.

θ	$r = 3\sin 2\theta$
0	$r = 3\sin(0) = 0$
$\pi/4$	$r = 3\sin(\pi/2) = 3$
$\pi/2$	$r = 3\sin(\pi) = 0$
$3\pi/4$	$r = 3\sin(3\pi/2) = -3$
$5\pi/4$	$r = 3\sin(5\pi/2) = 3$

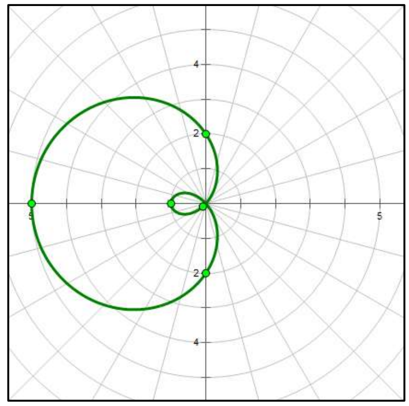


The change in selected values of θ in this problem are a result of the 2θ being the angle in the equation.

70) Graph $r = 2 - 3\cos\theta$.

This is a graph of a limaçon with an inner loop, symmetric about the x-axis, with most of the curve left of the y-axis.

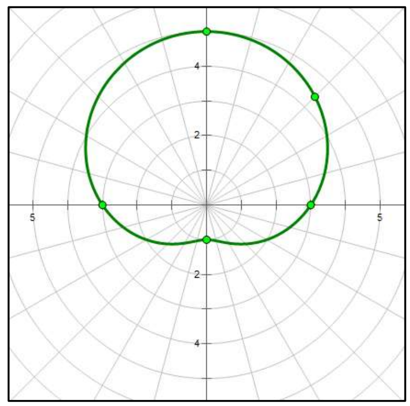
θ	$r = 2 - 3\cos\theta$
0	$r = 2 - 3\cos(0) = -1$
$\pi/4$	$r = 2 - 3\cos(\pi/4) \approx -0.12$
$\pi/2$	$r = 2 - 3\cos(\pi/2) = 2$
π	$r = 2 - 3\cos(\pi) = 5$
$3\pi/2$	$r = 2 - 3\cos(3\pi/2) = 2$



71) Graph $r = 3 + 2\sin\theta$.

This is a graph of a limaçon with a dimple, symmetric about the y-axis, with most of the curve above the x-axis.

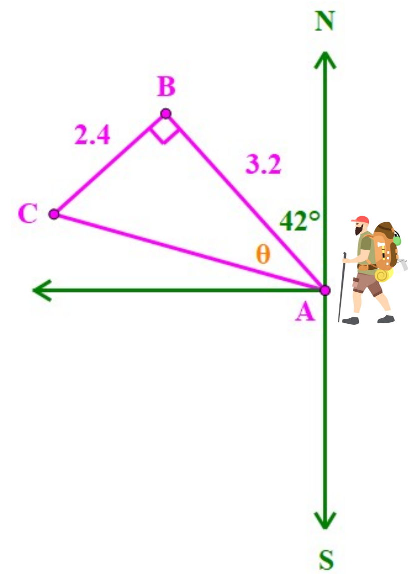
θ	$r = 3 + 2\sin\theta$
0	$r = 3 + 2\sin(0) = 3$
$\pi/4$	$r = 3 + 2\sin(\pi/4) \approx 4.41$
$\pi/2$	$r = 3 + 2\sin(\pi/2) = 5$
π	$r = 3 + 2\sin(\pi) = 3$
$3\pi/2$	$r = 3 + 2\sin(3\pi/2) = 1$



*72) A hiker leaves the trailhead with a bearing of N 42° W. After traveling 3.2 miles, the hiker then turns 90° and travels on a bearing of S 48° W for 2.4 miles. At that time, what is the bearing of the hiker from the trailhead? Round to the nearest tenth.

Using the diagram to the right:

- The hiker proceeds from Point A to Point B, then turns 90° left.
- The hiker then proceeds from Point B to Point C.
- We need to find the angle θ to determine the bearing.



$\tan\theta = \frac{2.4}{3.2} = 0.75$

$\theta = \tan^{-1} 0.75 \approx 36.9°$

Bearing angle from N to W is: $42° + 36.9° = 78.9°$

So, the bearing is: **N 78.9° W**

*73) A boat is in the ocean, and a nearby cliff has a restaurant on its edge. The angle of elevation from the boat to the top of the restaurant is 67 degrees, and the angle of elevation of the boat to the bottom of the restaurant is 48 degrees. If the boat is 50 feet away from the base of the cliff at sea level, then find the height of the restaurant, rounded to the nearest foot.

$\tan 48° = \frac{y}{50}$

$y = 50 \cdot \tan 48° = 55.5306 \text{ feet}$

$\tan 67° = \frac{x+y}{50}$

$x + y = 50 \cdot \tan 67° = 117.7926 \text{ feet}$

$x = 117.7926 - 55.5306 \approx$ **62 feet**